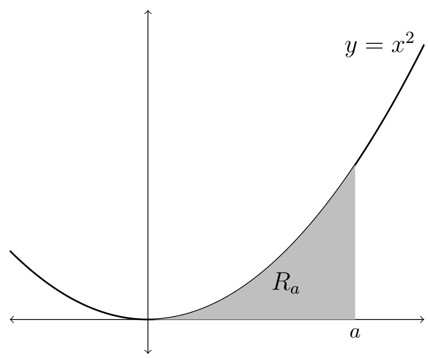
MATH 111, EXPLORATION 2

Due Friday, September 8

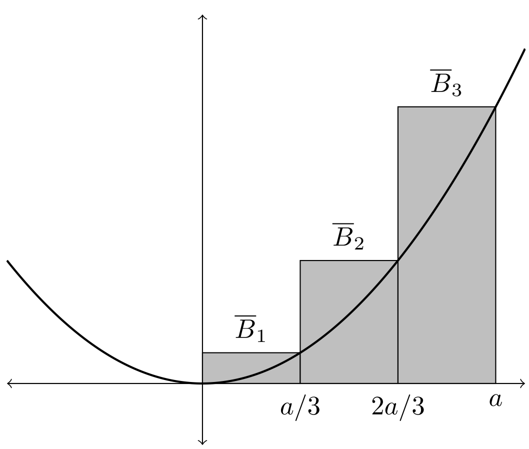
Your first task is an administrative one:

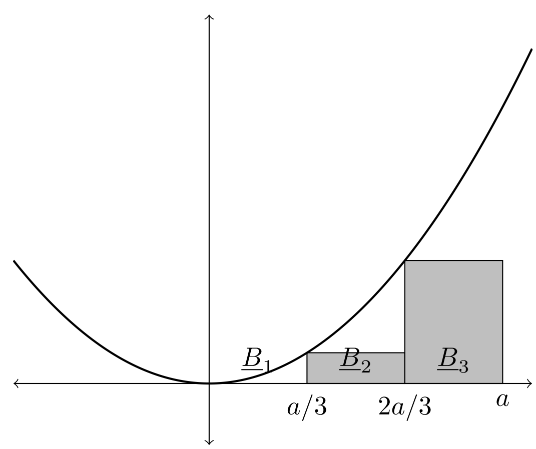
(1) Complete the "Introductory Science and Mathematics Student Survey"—you will receive an email with this subject line that includes a link to the survey.

In the next series of problems, you will compute the area of the region R_a in the plane bounded by the parabola $y = x^2$, the x-axis, the y-axis and the line $x = a$, where $a > 0$ is a fixed constant.



To this end, we will approximate the area by computing the area of a collection of boxes. Subdivide the interval $0 \leq x \leq a$ into 3 equal subintervals, and over each subinterval, draw a box $\overline{B}_i$ whose top lies entirely *over* the graph. Similarly, over each subinterval draw a box $\underline{B}_i$ whose top lies entirely *under* the graph.





In order to lie under the graph the first box on the right $\underline{B}_1$ must have height zero—that's why we can't see it in the picture. Notice that since the boxes $\overline{B}_1$, $\overline{B}_2$ and $\overline{B}_3$ contain the original region R_a, their area is an over-approximation of the area we wish to compute:

$$\text{area}(R_a) \leq \text{area}(\overline{B}_1) + \text{area}(\overline{B}_2) + \text{area}(\overline{B}_3) =: U_3$$

We let U_3 denote the over-approximation[1]. The letter "U" stands for *upper sum*. Similarly, the area of the boxes $\underline{B}_1$, $\underline{B}_2$, and $\underline{B}_3$ is an under-approximation of the area we wish to compute:

$$L_3 := \text{area}(\underline{B}_1) + \text{area}(\underline{B}_2) + \text{area}(\underline{B}_3) \leq \text{area}(R_a).$$

We let L_3 denote the under-approximation. The letter "L" stands for *lower sum.*

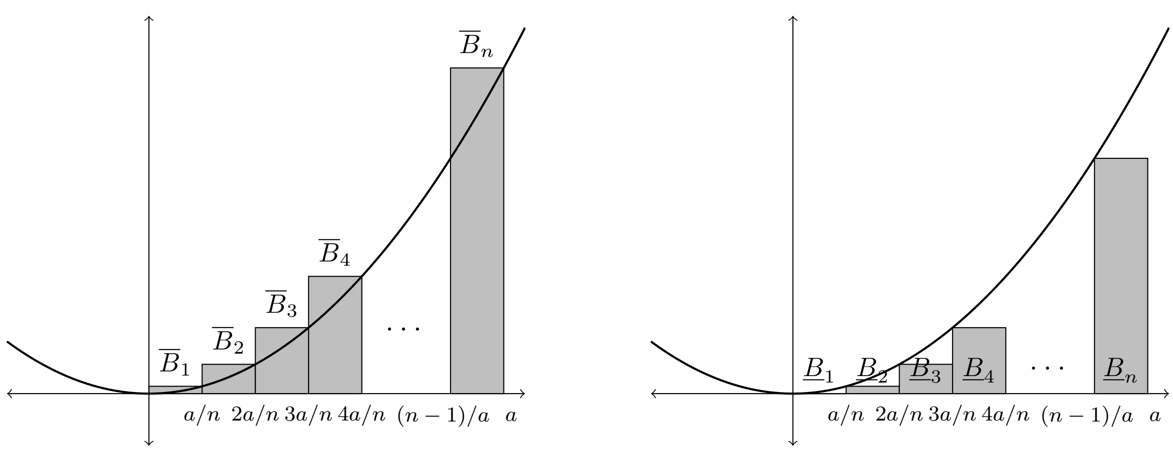
Let's compute the area of these boxes. For example, box $\overline{B}_1$ has width $\frac{a}{3}$ and, since its upper right corner lies on $y = x^2$, its height is $(a/3)^2 = a^2/9$. Thus,

$$\text{area}(\overline{B}_1) = \frac{a}{3} \cdot \frac{a^2}{9} = \frac{a^3}{27}.$$

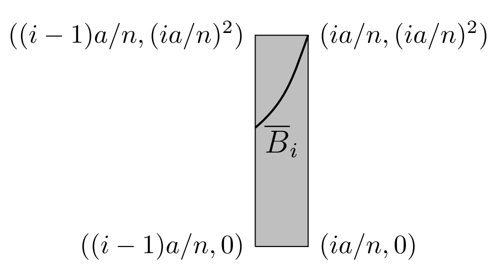
Now, perform a similar analysis of the area of each box to find the over and under approximations:

(2) Compute U_3. Your answer should be a formula depending only on a. Do the same for L_3. What are the over and under approximations when $a = 1$? When $a = 2$? When $a = 10$?

There was nothing particularly special about using 3 boxes to approximate the area of the region R_a. Let n denote an arbitrary positive integer (such as $n = 3$), and subdivide the interval $0 \leq x \leq a$ into n subintervals of equal width. For each subinterval, draw a box above and a box below the parabola $y = x^2$.



For each integer i, where $1 \leq i \leq n$, the i-th box $\overline{B}_i$ has coordinates



[1]The notation ":=" or "=:" means that the term on the left or right is defined by the equation

and so

$$\text{area}(\overline{B}_i) = \Big(\frac{ia}{n} - \frac{(i-1)a}{n}\Big)(ia/n)^2 = \frac{(i-(i-1))a}{n}\frac{i^2a^2}{n^2} = \frac{i^2a^3}{n^3}$$

Therefore, the over-approximation U_n of area(R_a) obtained by computing the area of the boxes $\overline{B}_1, \ldots, \overline{B}_n$ is

$$U_n := \text{area}(\overline{B}_1) + \cdots + \text{area}(\overline{B}_n) = \frac{a^3}{n^3} + \frac{4a^3}{n^3} + \frac{9a^3}{n^3} + \cdots + \frac{n^2a^3}{n^3}$$

There is a special notation[2] for taking an n-fold sum in which each term depends on an index i:

$$U_n = \frac{a^3}{n^3} + \frac{4a^3}{n^3} + \frac{9a^3}{n^3} + \cdots + \frac{n^2a^3}{n^3} = \sum_{i=1}^{n} \frac{i^2a^3}{n^3}.$$

The idea here is that i is an *index*, which runs from $i = 1$ to $i = n$. As it does, we add each new term to what we had before. Let's encode this into Mathematica so that we can calculate. The syntax for summation is:

`Sum[` (terms to be added, depending on i) `,{i,1,n}]`

Thus, the upper sum U_n can be written in Mathematica as

```
In[1]:= Sum[ (i^2 a^3)/n^3, {i, 1, n}]
```

We want to compute this for different values of n and a, so let's make a function $U(a, n)$ of both of these variables that computes the over-approximation of the region R_a using n boxes:

```
In[6]:= U[a_, n_] := Sum[ (i^2 a^3)/n^3, {i, 1, n}]
```

We can now ask Mathematica to evaluate the over-approximation for particular values of n and a:

```
In[7]:= U[a, 10]
Out[7]= 77 a^3/200

In[9]:= U[2, 10]
Out[9]= 77/25

In[10]:= U[2, 10] // N
Out[10]= 3.08
```

To make Mathematica display an answer numerically, in terms of decimals, type `//N` after the entry.

(3) Check your computation in (2) by evaluating $U(a, 3)$ in Mathematica for $a = 1, 2, 10$.

(4) Compute area($\underline{B}_i$) and then write down a formula for the under-approximation L_n. Encode this into Mathematica as a function $L(a, n)$. Check your computation in (2) for $L(a, 3)$ when $a = 1, 2, 10$.

[2] The symbol Σ is a capital Greek *sigma* and stands for "sum".

(5) Evaluate $U(1, n)$ and $L(1, n)$ for $n = 10, 100, 1000$, and 10000. What is area(R_1)?

(6) Do the same for $U(2, n)$ and $L(2, n)$. What is area(R_2)? How about area(R_{10})?

(7) What is area(R_a) in general?

To justify your answer to (7), you might want to use the following formula for the sum of the squares of the first n integers (which you may have already discovered that Mathematica knows about!):

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6}.$$

There is a similar formula for the sum of the first n integers:

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2}.$$

Here is a challenge problem:

(8) Why are these formulas true?