LIMITS: the joy of ϵ and δ

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Here is the rigorous definition of the limit.

Definition. *The statement* $\lim_{x \to c} f(x) = L$ *means that for every* $\epsilon > 0$*, there exists* $\delta > 0$ *such that*

$$\textit{if } 0 < |x - c| < \delta, \quad \textit{then } |f(x) - L| < \epsilon \; .$$

Remember that

$$\begin{aligned} |f(x) - L| < \epsilon & \qquad \text{means that } f(x) \text{ is within } \epsilon \text{ of } L \text{, and} \\ 0 < |x - c| < \delta & \qquad \text{means that } x \text{ is within } \delta \text{ of } c \text{ and } x \neq c. \end{aligned}$$

The number ϵ is an arbitrary degree of closeness to the limit L, and it determines the target region that we want the function $f(x)$ to land in. We must find a number δ, which represents a degree of closeness to c that we have control over, so that

$$\text{if } x \text{ is within } \delta \text{ of } c \text{ (and } x \neq c\text{)}, \quad \text{then } f(x) \text{ is within } \epsilon \text{ of } L.$$

This captures in a precise way the idea that $f(x)$ can be made arbitrarily close to L by plugging in values of x that are arbitrarily close to c.

The number δ depends on ϵ, and in proofs it will almost always be a formula involving ϵ. Here is a general strategy to follow in doing preparatory scratchwork and writing an ϵ-δ proof.

STRATEGY

(I) Examine $|f(x) - L|$ and manipulate it algebraically so that it looks like

$$|f(x) - L| = \cdots = (\text{something})|x - c|.$$

We have control over the number $|x - c|$, because in the course of the proofs we assume that it is less than δ. In other words, we can make it as small as we want.

(II) Next, examine the expression (something) from step (I), manipulate it, and possibly make assumptions on δ so that

$$(\text{something}) \leq A, \qquad \text{for some positive real number } A.$$

This step may require some creativity, and will be different depending on the form of the function.

(III) Arrange so that $\delta \leq \epsilon/A$. For example, if you assumed that $\delta \leq B$ in step (II) in order to use $|x-c| < B$ in deriving the inequality, then you could set $\delta = \min\{B, \epsilon/A\}$. This insures that $\delta \leq B$ and $\delta \leq \epsilon/A$, so that your previous work will allow you to deduce that

$$\begin{aligned} |f(x) - L| = (\text{something})|x - c| &\leq A \cdot |x - c| \\ &< A \cdot \delta \\ &\leq A \cdot \frac{\epsilon}{A} = \epsilon. \end{aligned}$$

(IV) Now that the scratchwork is done, set it aside and write down your proof on a new sheet of paper. It should have the following general format:

Proof. Let $\epsilon > 0$. Set $\delta = \ldots$. Assume that $0 < |x - c| < \delta$. ... Then:

$$\begin{aligned} |f(x) - L| &= \cdots \\ &< \cdots \\ &\vdots \\ &= \epsilon. \end{aligned}$$

□

Here are some useful facts that can help when manipulating formulae in ϵ-δ proofs.

- $|a \cdot b| = |a||b|$
- $\left|\frac{a}{b}\right| = \frac{|a|}{|b|}$
- if $a \geq b$, then $1/a \leq 1/b$.
- if $a \leq b$, then $a + c \leq b + c$.
- if $a \leq b$ and $c > 0$, then $ca \leq cb$.
- if $a \leq b$ and $c < 0$, then $ca \geq cb$.
- $|a + b| \leq |a| + |b|$ (the triangle inequality)
- $\big||a| - |b|\big| \leq |a - b|$ (the reverse triangle inequality)
- $a^2 - b^2 = (a + b)(a - b)$, for instance $a - b = (\sqrt{a} + \sqrt{b})(\sqrt{a} - \sqrt{b})$.

EXERCISES

For (1) and (2), fill in the blanks to complete the proofs.

1. Claim: $\lim_{x \to 2}(x^2 - x + 3) = 5$.

   Proof. Let $\epsilon > 0$ be given. Set $\delta = \min\{1, \_\_\_\_\_\}$. Suppose that $0 < |x - \_\_\_\_\_| < \delta$. Then:

$$
\begin{aligned}
|(x^2 - x + 3) - \_\_\_\_\_| &= |x^2 - x - 2| \\
&= |x + 1||x - 2| \\
&= |(x - 2) + \_\_\_\_\_\_\_||x - 2| \quad \text{(by algebraic manipulation)} \\
&\leq \big(|x - 2| + \_\_\_\_\_\_\_\big)|x - 2| \quad \text{(by the triangle inequality)} \\
&< \big(\delta + \_\_\_\_\_\_\_\big)\delta \quad \text{(by assumption)} \\
&\leq \big(1 + \_\_\_\_\_\_\_\big) \cdot (\epsilon/\_\_\_\_\_\_\_) \quad \text{(because } \delta < 1 \text{ and } \delta < \_\_\_\_\_\_\text{)} \\
&= \_\_\_\_\_\_ \cdot (\epsilon/\_\_\_\_\_\_\_) \\
&= \epsilon.
\end{aligned}
$$

$\square$

2. Let $f(x) = \sqrt{x+1}$. Claim: $\lim_{x \to 1} f(x) = \sqrt{2}$.

Proof. Let $\epsilon > 0$. We choose $\delta = \epsilon\sqrt{2}$. Suppose that $0 < |x-1| < \delta$. Since the square-root function always gives non-negative values as output,

$$\sqrt{x+1} \geq 0.$$

Adding __________ to each side, we get

$$\sqrt{x+1} + \sqrt{2} \geq \sqrt{2}.$$

Be taking reciprocals, we have the new inequality

$$\frac{1}{\sqrt{x+1}+\sqrt{2}} \leq \frac{1}{\sqrt{2}}. \tag{$\star$}$$

Now consider $|f(x) - \sqrt{2}|$, which we must show is less than ϵ. By multiplying the denominator and the numerator by __________________________, we see that:

$$|\sqrt{x+1} - \sqrt{2}| = \frac{|\sqrt{x+1}-\sqrt{2}||\sqrt{x+1}+\sqrt{2}|}{|\sqrt{x+1}+\sqrt{2}|}$$

$$= \frac{1}{|\sqrt{x+1}+\sqrt{2}|} \cdot |x+1-2|$$

$$= \frac{1}{|\sqrt{x+1}+\sqrt{2}|} \cdot |x-1|$$

$$< \frac{1}{|\sqrt{x+1}+\sqrt{2}|} \cdot \delta \quad \text{(by \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_)}$$

$$\leq \frac{1}{\sqrt{2}}\delta \quad \text{(by \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_)}$$

$$= \frac{1}{\sqrt{2}}(\epsilon\sqrt{2}) \quad \text{(by \_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_\_)}$$

$$= \epsilon.$$

□

For (3)–(6), Find the indicated limit, then prove that your claimed limit is correct using the rigorous definition of the limit.

3. $\lim_{x \to -4} (-2x + 1)$

4. $\lim_{x \to 2} (3x^2 - 4x - 3)$

5. $\lim_{x \to 9} \sqrt{x}$

6. $\lim_{x \to 1} \frac{1}{x - 2}$

The remaining exercises are an opportunity for further exploration.

7. Use the rigorous definition of the limit to prove that if a and b are any real numbers, then $\lim_{x \to c} (ax + b) = ac + b$.

8. Consider the function $f(x) = \frac{|x|}{x}$. Use the rigorous definition of right/left limits from class to prove that

$$\lim_{x \to 0^+} f(x) = 1 \quad \text{and} \quad \lim_{x \to 0^-} f(x) = -1.$$

   Where is the function f continuous?

9. Write down a rigorous definition of the statement $\lim_{x \to +\infty} f(x) = L$. Use your definition to prove that $\lim_{x \to +\infty} \frac{2x}{x - 1} = 2$.