Sandpiles and Synthesizers: Listening to the Discrete Laplacian

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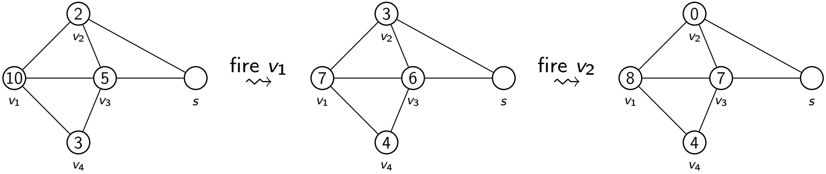
The Abelian Sandpile Model

Start with a graph and a number of "grains of sand" at each vertex.

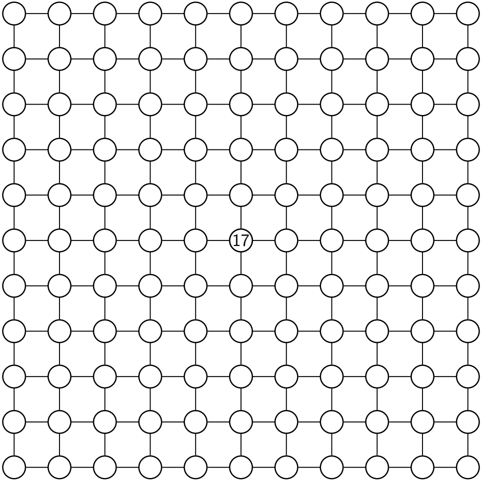
To *fire* a vertex, send one grain of sand along each incident edge.



The sink vertices make sand disappear when it is sent to them.

The final configuration does not depend on the order of firings—hence the term *abelian* sandpile model.

Starting with 17 grains of sand in the middle of an 11x11 grid:



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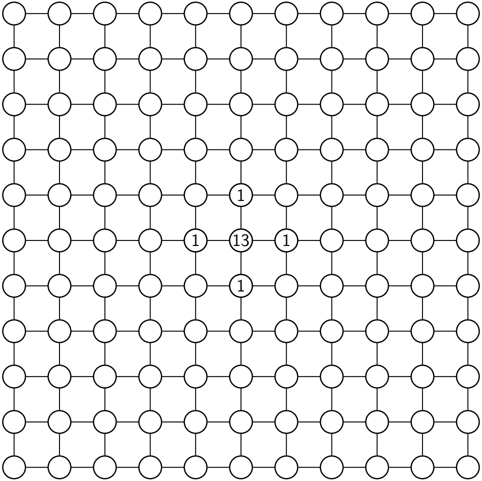

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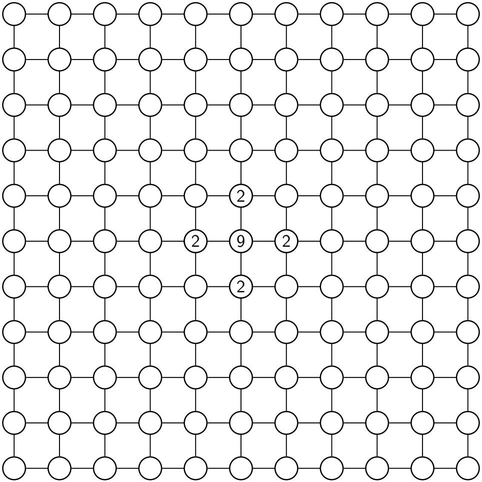

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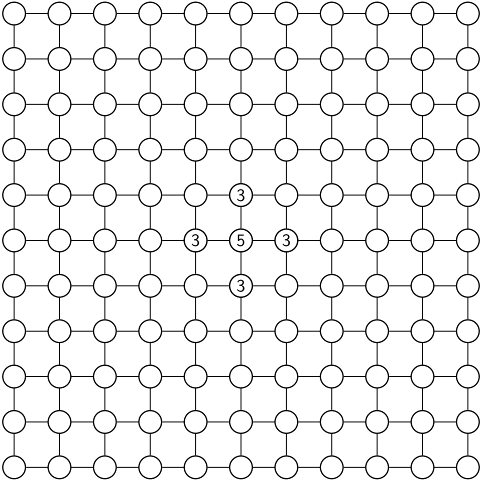

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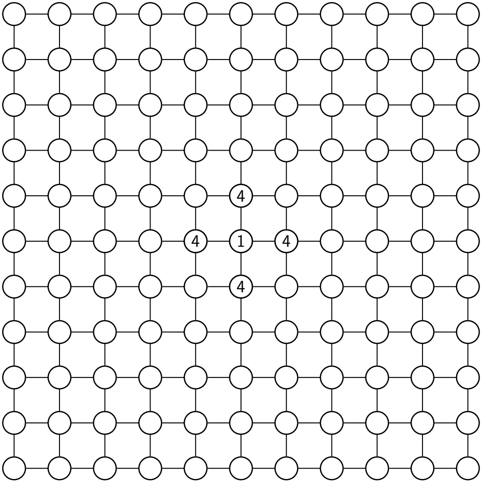

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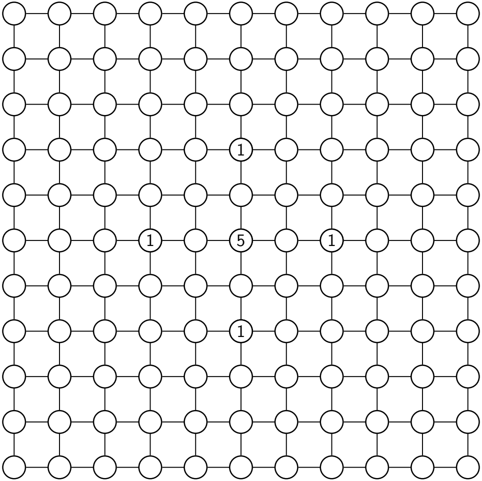

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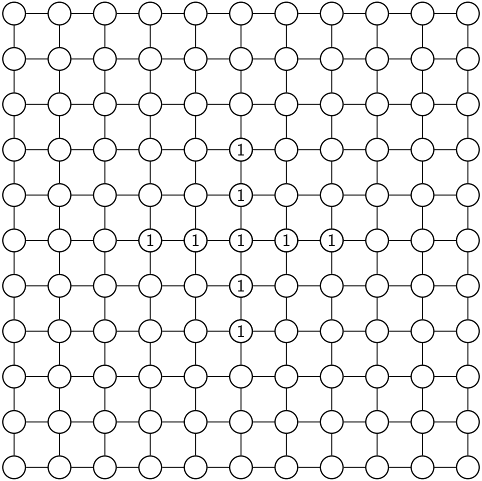
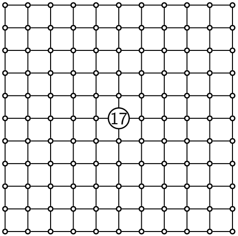
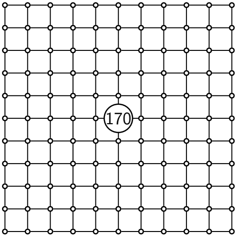
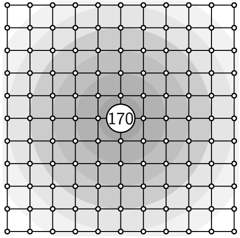


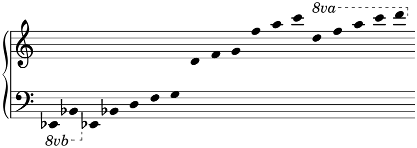
Great, let's make some music out of this!







Random volume and note length.
Note pitches randomly selected from:



Same, except:
note pitches descend
from center of grid

A word from our sponsors:

The possible configurations of sand on a graph are elements of the free abelian group $\mathbf{Z}V$ on the set of vertices. The isomorphism

$$\mathbf{Z}V \xrightarrow{\cong} \mathbf{Z}^V$$

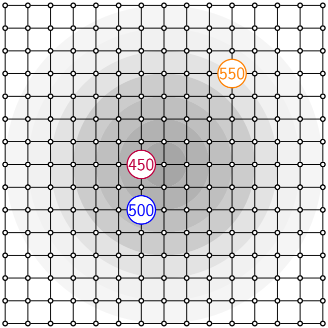
$$v \longmapsto \chi_v, \quad \text{where} \quad \chi_v(w) = \begin{cases} 1 & \text{if } w = v, \\ 0 & \text{else} \end{cases}$$

is a combinatorial analogue of the duality between divisors and meromorphic functions on a Riemann surface.

The abelian sandpile model is a discrete model for the flow of heat. Both types of processes are regulated by a *Laplace operator* $\Delta\colon \mathbf{Z}^V \longrightarrow \mathbf{Z}^V$

$$(\Delta\sigma)(v) = \sum_{\substack{\text{edges from} \\ v \text{ to } w}} (\sigma(v) - \sigma(w)) \quad \longleftrightarrow \quad \Delta\sigma = \sum_i \frac{\partial^2 \sigma}{\partial x_i^2}$$

IV-V-I progression on a 15 x 15 grid

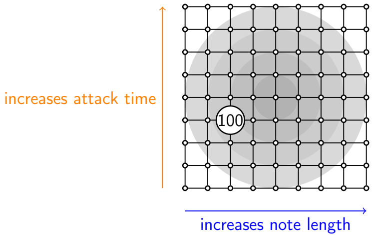


IV chord: at time $t = 0$

V chord: at time $t = 100$

I chord: at time $t = 200$

Note length and attack variation on a 9 x 9 grid



Same notes on bullseye as before.

Longer piece on a 41 x 41 grid