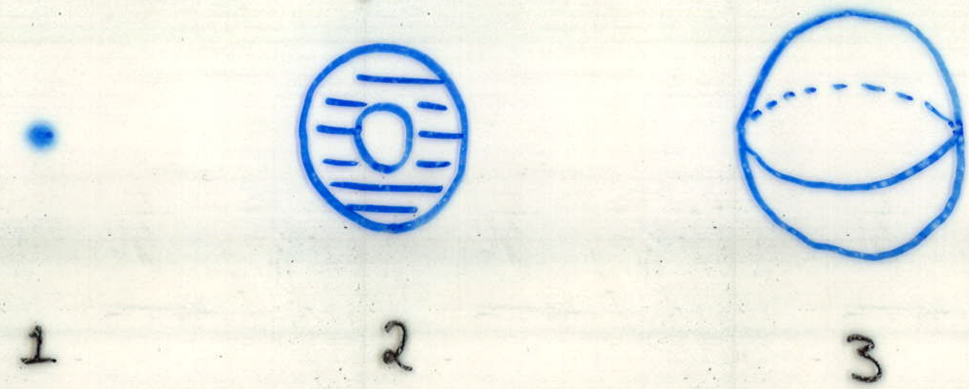
WHAT ARE NUMBERS?

> Numbers are the structure that counts how many things are there.



3 things!

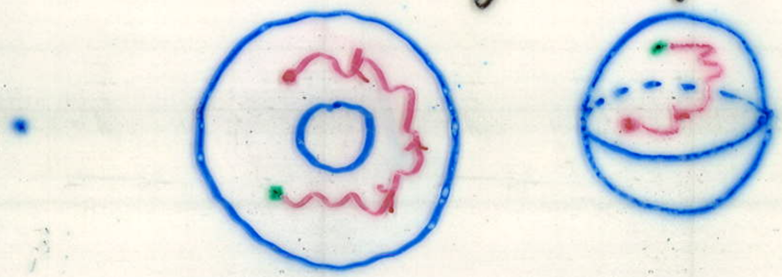
Perhaps this is a definition of how we _use_ numbers, not _what_ they are.

FREGE:

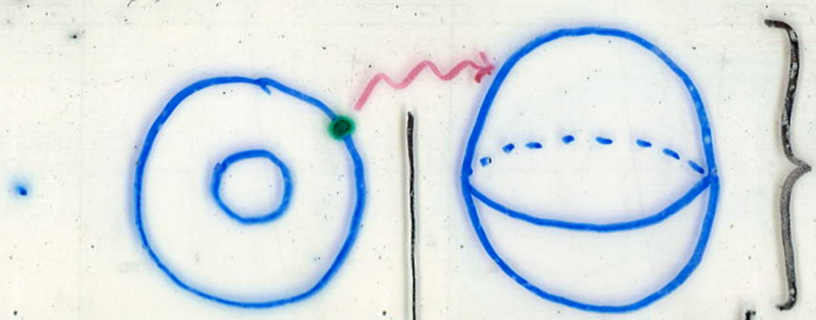
> a number is a totality of all collections of objects that are in correspondence with each other.

OK, SO HOW DO WE COUNT?

IDEA: use a point as a "test object" to distinguish the different components within a collection of things.



- PLACE A POINT DOWN
- CONSIDER ALL OF THE OTHER POINTS THAT YOU CAN "WIGGLE" TO, i.e. CONSIDER PATHS
- THE <u>COMPONENT</u> DETERMINED BY THE INITIAL POINT IS ALL OF THE PLACES YOU CAN FIND A PATH TO.



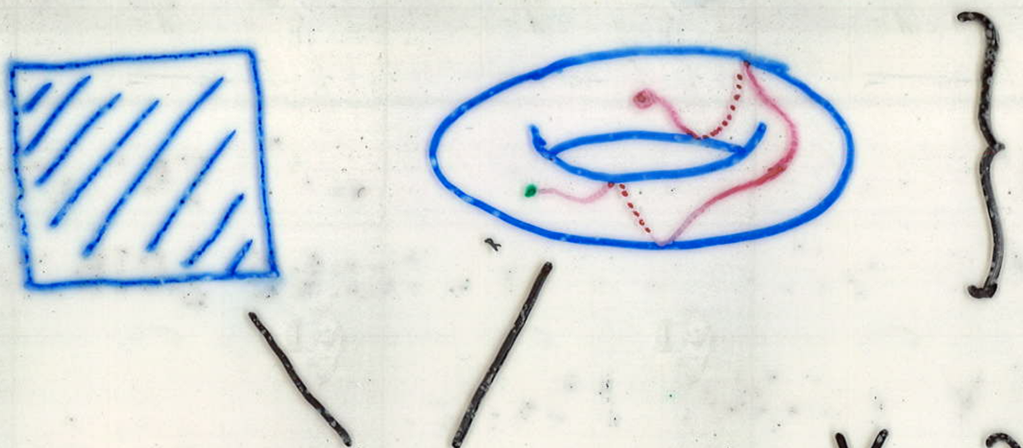
3 DIFFERENT COMPONENTS

THERE IS <u>NO</u> PATH B/W THE WASHER & SPHERE

THE COMPONENTS COUNT THE NUMBER OF THINGS THAT WE HAVE.

This invariant is determined by "wiggling" a point · around.

wiggle = continuous deformation = homotopy



X

2 components: $\pi_0 X = 2$

DEFINE

$\pi_0 X$ = THE DIFFERENT WAYS OF PUTTING · IN X, UP TO DEFORMATION

= THE NUMBER OF THINGS IN X

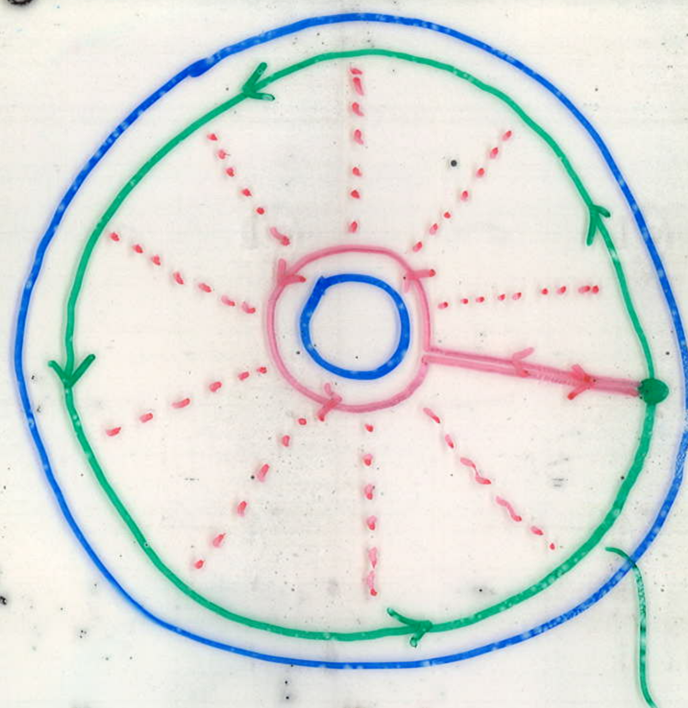
the process $X \longmapsto \pi_0 X$ LOSES INFORMATION!

$\pi_0\Big($  $\Big) = 1$ and $\pi_0\Big($  $\Big) = 1$

but the washer and disk are different!

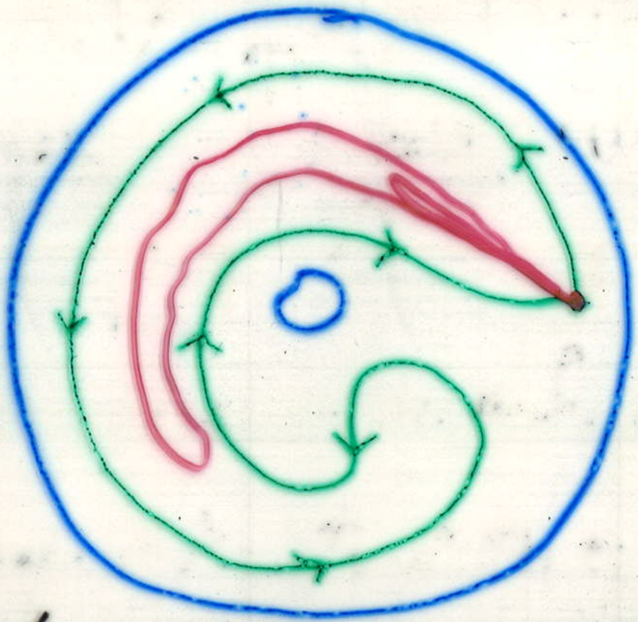
In order to detect the hole in the washer, use  as a "test object", instead of .

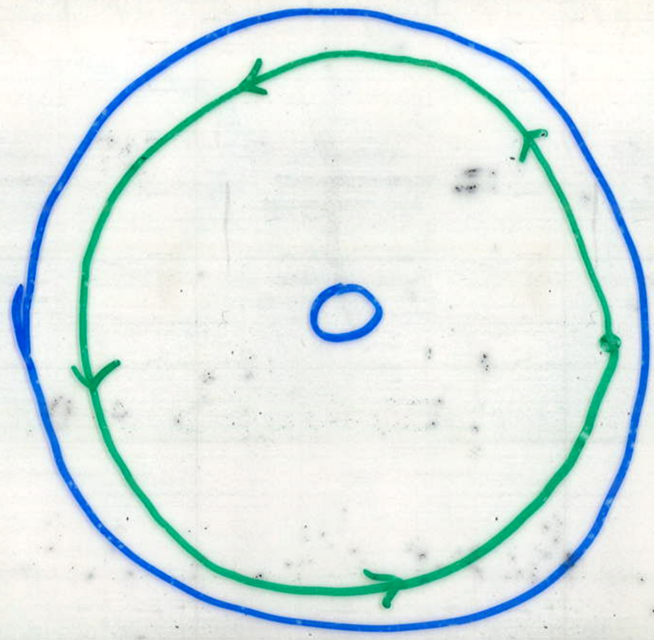
PLACE A LOOP DOWN.
CONSIDER ALL OF THE OTHER LOOPS WE <u>CAN</u> DEFORM IT TO.



ALL OF THE LOOPS "WIGGLED" FROM α GO <u>ONCE</u> AROUND THE HOLE (IN THE COUNTER CLOCKWISE DIRECTION)

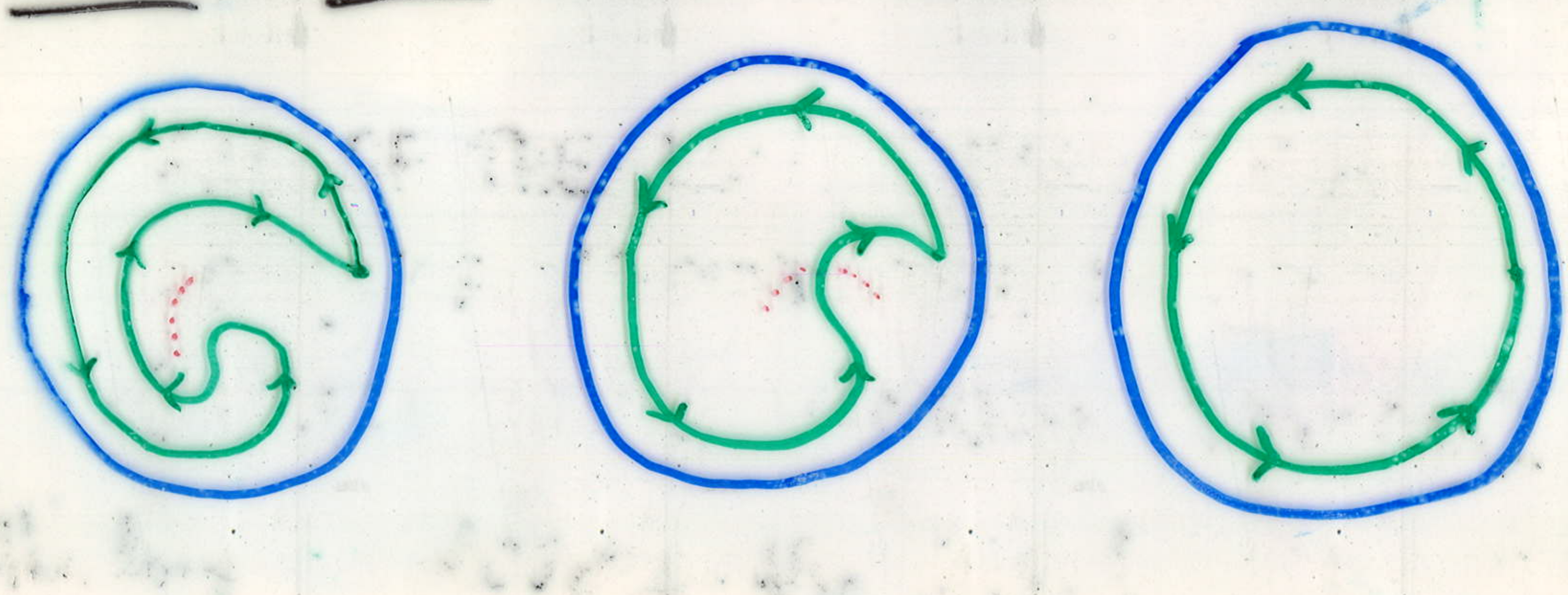
the loop α detects the hole!





THIS LOOP IS DIFFERENT FROM α
B/C IT CAN BE DEFORMED TO THE
TRIVIAL LOOP • (A SINGLE POINT).

BUT IN THE DISK, THEY ARE THE
<u>SAME</u> <u>LOOP</u> :

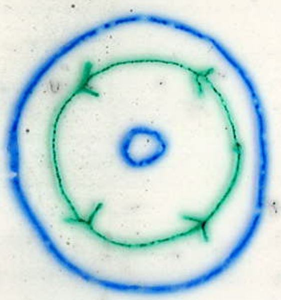


$\pi_1 X$ = THE SET OF DIFFERENT LOOPS IN X, UP TO DEFORMATION.

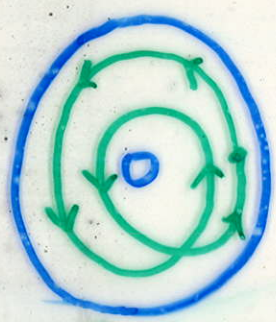
$\pi_1\Big($  $\Big) = \{ \cdot \}$

ALL LOOPS IN THE DISK CAN BE DEFORMED TO A POINT $\cdot$.

$\pi_1\Big($  $\Big) = \{ \cdot, \alpha, 2\alpha, 3\alpha, 4\alpha, \ldots, -\alpha, -2\alpha, \ldots \}$

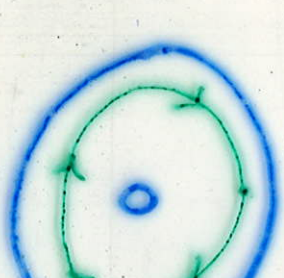


α



2α

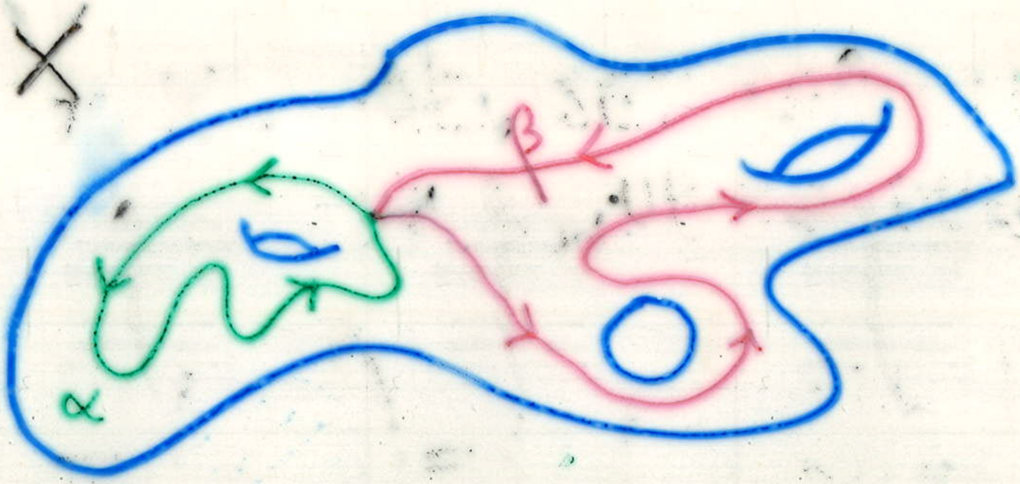
$\ldots$ $n\alpha$ = loop that goes around the hole n times (in CCW direction)



$-\alpha$

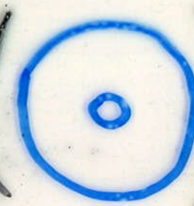
$\pi_1(\;)$ distinguishes between washer and disk

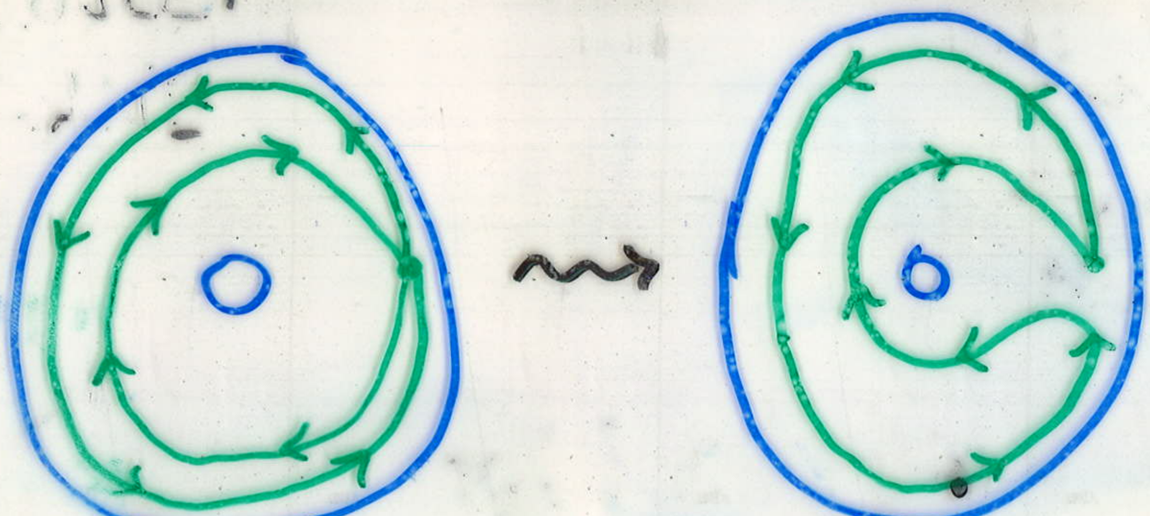
X



If α, β are two loops in X, then
$\alpha \boxplus \beta$ is the loop "do α, then β".

THE OPERATION $\boxplus$ MAKES $\pi_1 X$ INTO A <u>GROUP</u> (= a set with an operation like + of numbers).

$\alpha \boxplus -\alpha = \cdot$ in π_1()



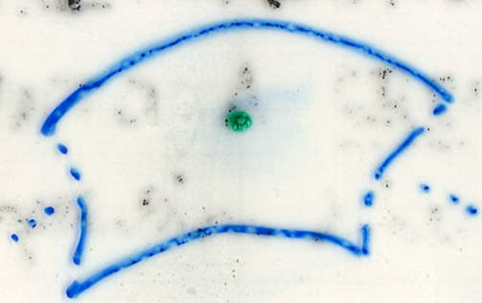
$\alpha \boxplus -\alpha$ $\simeq \cdot$

The operation $\boxplus$ on loops need not behave like $+$ of numbers in all ways.

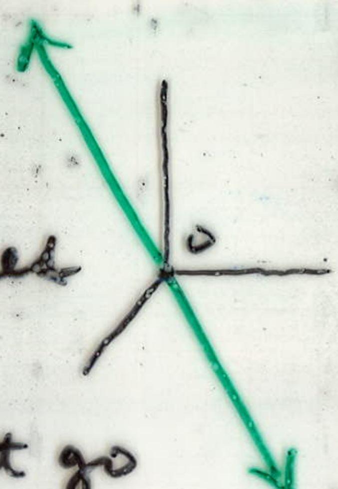
$\alpha \boxplus \beta \neq \beta \boxplus \alpha$ in general!

<u>an example where $\alpha \boxplus \alpha = \cdot$</u>

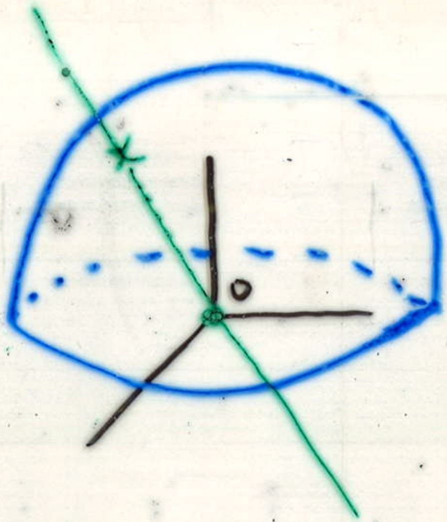
PROJECTIVE PLANE = SPACE OF LINES IN 3 DIM'L SPACE



points $\cdot$ in PROJECTIVE PLANE $\longleftrightarrow$ lines

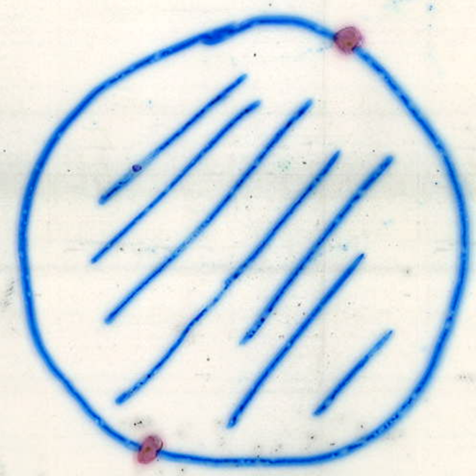


that go through a fixed point O.



LINE IS DETERMINED BY WHERE IT INTERSECTS NORTHERN HEMISPHERE

"UNFOLD DOME, IDENTIFY OPPOSITE POINTS ON EQUATOR"

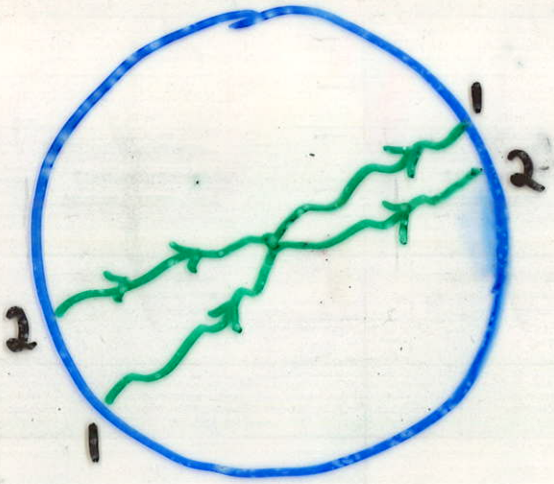


identify • $\rightsquigarrow$ •

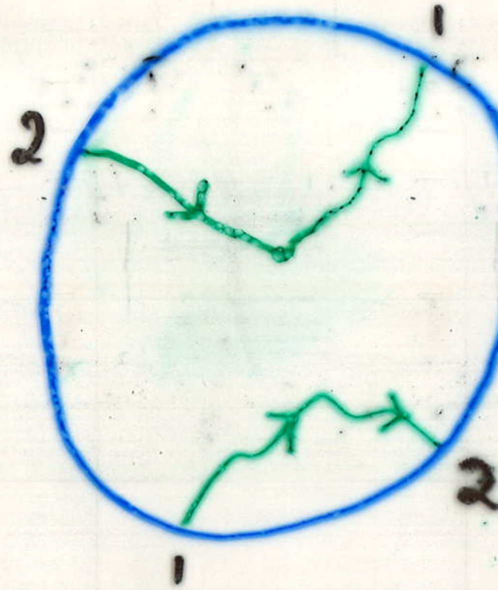
α

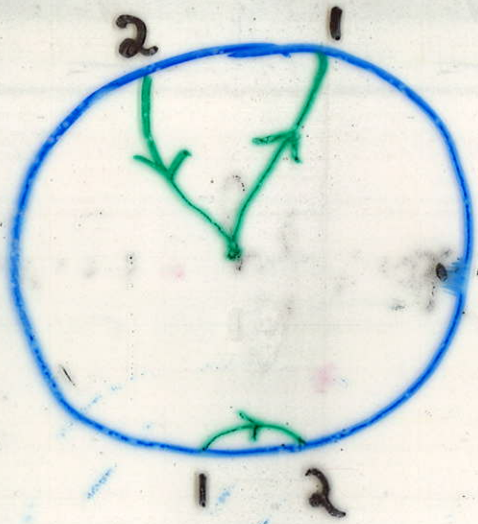
loop in projective plane,

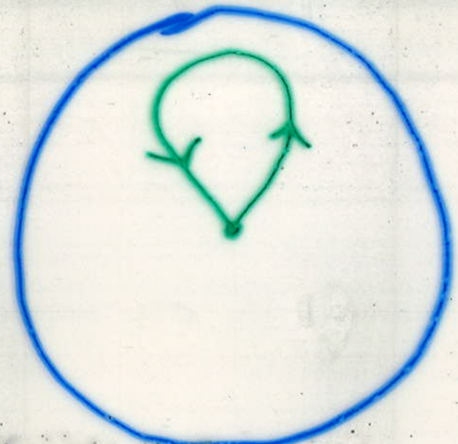
$\alpha \neq \cdot$

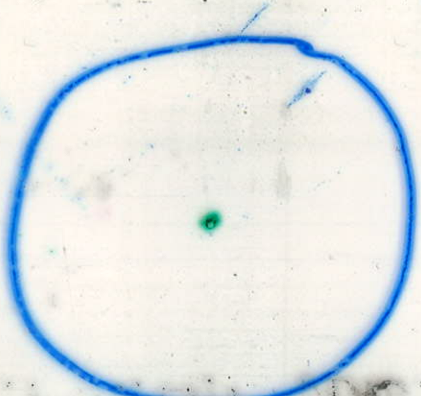


$\alpha \boxplus \alpha$



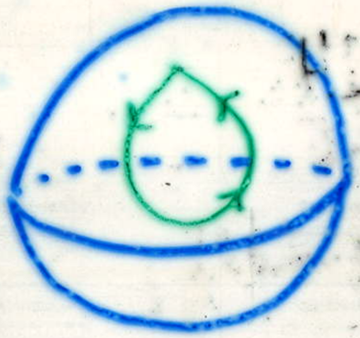






SO: $\alpha \boxplus \alpha = \cdot$

$\pi_1\left(\begin{array}{c}\text{PROJECTIVE}\\ \text{PLANE}\end{array}\right) = \{\cdot, \alpha\}$



ANY LOOP ON THE SPHERE CAN BE DEFORMED TO A POINT • !!!

$\pi_1\Big($  $\Big) = \{ \cdot \}$,

we need a new test object to detect the higher dimensional hole in the sphere!

instead of • or , we will use .

$\pi_2 X =$ THE DIFFERENT WAYS OF PLACING  IN X, UP TO DEFORMATION

α $\qquad\qquad$ $\alpha \boxed{+} \alpha = 2\alpha$

$\pi_2\Big($  $\Big) = \{ \cdot, \pm\alpha, \pm 2\alpha, \pm 3\alpha, \ldots \}$

$X \mapsto \pi_0 X, \pi_1 X, \pi_2 X, \ldots$

GIVES A COMPLETE SYSTEM OF TOPOLOGICAL INVARIANTS:

$\pi_n X = \pi_n Y$ for all $n \geq 0$ $\iff$ X and Y are TOPOLOGICALLY EQUIVALENT (DEFORMABLE INTO EACH OTHER)

$\pi_0 X$ = THE DIFFERENT WAYS OF PLACING • IN X, UP TO DEFORMATION

= THE NUMBER OF THINGS IN X

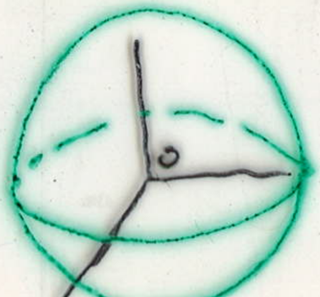
$\pi_1 X$ = THE DIFFERENT WAYS OF PLACING  IN X, UP TO DEFORMATION

$\pi_2 X$ = THE DIFFERENT WAYS OF PLACING  IN X, UP TO DEFORMATION

$\vdots$

$\pi_n X$ = THE DIFFERENT WAYS OF PLACING THE n-DIMENSIONAL SPHERE IN X, UP TO DEFORMATION.

n-SPHERE = $\{$ POINTS IN (n+1)-DIM SPACE THAT HAVE DISTANCE 1 FROM 0 $\}$



2-SPHERE

LET n-WORLD = SPACE WITH:

FOR EACH COLLECTION $x = \{x_1, \ldots, x_n\}$ OF n THINGS

FOR EACH PERMUTATION THAT REARRANGES $x = \{x_1, \ldots, x_n\}$ INTO $y = \{y_1, \ldots, y_n\}$

FOR EACH COMPARISON $H = G \circ F$ BETWEEN THE PERMUTATIONS F, G, H

FOR EACH COMPARISON B/W COMPARISONS B/W...

BARRATT-PRIDDY-QUILLEN <u>THEOREM</u>:

$$\pi_n\left(\begin{array}{c}\text{ADD NEGATIVES TO} \\ \text{0-WORLD, 1-WORLD, 2-WORLD,...}\end{array}\right) = \lim_{k \to \infty} \pi_{k+n}\left(\text{k-SPHERE}\right)$$