DUALITY SEMINAR

These are notes from a seminar at Regensburg organized by John Lind and Oriol Raventós-Morera in the spring of 2015. Our aim was to understand the paper "Duality in algebra and topology" by Dwer-Greenlees-Iyengar.

1. Context and smallness conditions

Our setting is the category Mod_S of modules over the sphere spectrum S. DGI choose to realize this as the category of symmetric spectra equipped with the Hovey-Shipley-Smith model structure, but we will work independent of a choice of model as much as possible. The ∞-categorically minded may prefer to work in that context, with the warning that it is their responsibility to make meaningful statements. The category Mod_S is closed symmetric monoidal under the smash product $\otimes = \otimes_S$ and the function spectrum functor $\mathrm{Hom}(-,-) = \mathrm{Hom}_S(-,-)$. Often, we implicitly state conclusions in the stable homotopy category $\mathrm{ho}\,\mathrm{Mod}_S$, but as in the article, we will not be too precise in moving between Mod_S and $\mathrm{ho}\,\mathrm{Mod}_S$. In particular, whenever we use a functor, we will mean the derived version of that functor, and will not be explicit about cofibrant/fibrant approximation, etc.

A (commutative) S-algebra is a (commutative) monoid in Mod_S under $\otimes$. If R is an S-algebra, we may form the smash product $M \otimes_R N$ of a right R-module M and a left R-module N. The category of R-modules is enriched in Mod_S via the function spectrum of R-module maps $\mathrm{Hom}_R(-,-)$. Note that unless R is commutative, $M \otimes_R N$ and $\mathrm{Hom}_R(M,N)$ need not be R-modules.

We have the Eilenberg-MacLane embedding $R \longmapsto HR$ of discrete rings into S-algebras. We have an equivalence $D(R) \simeq \mathrm{ho}\, H\,\mathrm{Mod}_R$ of the derived category of R-modules with the homotopy category of HR-modules, realized by the functor taking a chain complex of R-modules C to the HR-module HC with n-th space

$$HC_n = N^{-1}\tau_{\geq 0}\Sigma^n C,$$

where N^{-1} is the inverse of the Dold-Kan correspondence $N\colon \mathrm{sAb} \to \mathrm{Ch}^+$. From now on we will follow the algebraic notation and write $D(R)$ for the homotopy category of modules over an S-algebra R.

We will say that an S-module is small if it is compact as an object of the triangulated category $\mathrm{ho}\,\mathrm{Mod}_S$, i.e. if the functor $\mathrm{Hom}(X,-)$ commutes with coproducts. X is small if and only if it is dualizable as an object of the closed symmetric monoidal category $\mathrm{ho}\,\mathrm{Mod}_S$, i.e. if the natural map

$$\mathrm{Hom}(X,S) \otimes Y \to \mathrm{Hom}(X,Y)$$

is an equivalence for all S-modules Y. This in turn is equivalent to the condition that X is a finite spectrum.

We say that a subcategory of a triangulated category is a thick subcategory if it is closed under equivalences, cofiber sequences, and retracts. If it is additionally closed under coproducts (and thus all colimits) we say that it is a localizing subcategory.

Definition 1.1. Let R be an S-algebra, and let A and B be R-modules. The R-module A is *finitely built* from B if A is contained in the smallest thick subcategory containing B. We say that A is *built* from B if k is contained in the smallest localizing subcategory, containing B.

Lemma 1.2. *An R-module k is small (=compact) in* Mod_R *if and only if k is finitely built from R.*

Proof. It is actually a general fact that in any compactly generated triangulated category, the compact objects are precisely the objects of the smallest thick subcategory containing the generators, and our proof can be adapted to work in this generality. Notice that the R-modules X satisfying the condition

$$\mathrm{Hom}_R(X, \amalg Y_i) \simeq \amalg \mathrm{Hom}_R(X, Y_i)$$

form a thick subcategory. Since R itself is small, it follows that the smallest thick subcategory containing R is contained in the subcategory of small R-modules. This gives one implication. For the other, assume that k is small. We may write k as a colimit of R-modules finitely built from R

$$k \simeq \operatorname{colim} k_\alpha,$$

say by taking a cofibrant approximation as a cell R-module and forming the colimit of the finite subcomplexes. (More generally, the fact that Mod_R is compactly generated means that the smallest localizing subcategory containing the generator R is all of Mod_R, i.e. every R-module is built from R.) Since k is small, we have equivalences

$$\mathrm{Hom}_R(k, k) \simeq \mathrm{Hom}_R(k, \operatorname{colim} k_\alpha) \simeq \operatorname{colim} \mathrm{Hom}_R(k, k_\alpha).$$

Therefore the identity map id_k factors through some stage of the colimit k_α, so k is a retract of a finitely built R-module. Hence k itself is finitely built from R. □

We now fix an S-algebra R, an R-module k, and define $\mathcal{E} = \mathrm{End}_R(k)$.

Definition 1.3. A map $f\colon U \longrightarrow V$ of R-modules is a k-equivalence if $\mathrm{Hom}_R(k, f)$ is an equivalence. An R-module S is k-cellular(= k-torsion) if $\mathrm{Hom}_R(X, f)$ is an equivalence for all k-equivalences f.

Lemma 1.4. *A k-equivalence $f\colon X \longrightarrow Y$ of k-cellular R-modules is an equivalence.*

Proof. The equivalence

$$\mathrm{Hom}_R(Y, f)\colon\ \mathrm{Hom}_R(Y, X) \xrightarrow{\simeq} \mathrm{Hom}_R(Y, Y)$$

furnishes a map $g\colon Y \longrightarrow X$ such that $f \circ g \simeq \mathrm{id}_Y$. The image of $g \circ f$ under the equivalence

$$\mathrm{Hom}_R(X, f)\colon\ \mathrm{Hom}_R(X, X) \xrightarrow{\simeq} \mathrm{Hom}_R(X, Y)$$

is $f \circ g \circ f \simeq f$, from which we deduce that $g \circ f \simeq \mathrm{id}_X$. □

Theorem 1.5. *For each R-module X, we may functorially construct a k-equivalence* $\mathrm{Cell}_k X \to X$ *where* $\mathrm{Cell}_k X$ *is a k-cellular R-module. We refer to* $\mathrm{Cell}_k X$ *as a k-cellular approximation to X.*

Let $\mathrm{Cell}_k D(R)$ denote the full subcategory of $D(R)$ spanned by the k-cellular objects. The theorem gives the right adjoint of the adunction

$$\mathrm{Cell}_k D(R) \underset{\mathrm{Cell}_k(-)}{\overset{\text{inclusion}}{\rightleftarrows}} D(R)$$

which exhibits the k-cellular R-modules as a right Bousfield localization (= colocalization = cellularization) of the homotopy category of R-modules.

Proposition 1.6. *An R-module X is k-cellular if and only if X is built from k.*

The proof is an explicit construction of $\mathrm{Cell}_k X$ via the small object argument that shows that it is built from k.

The R-module k is also a (left) $\mathcal{E} = \mathrm{End}_R(k)$-module, and the actions commute. Given an $\mathcal{E}^{\mathrm{op}}$-module X, the smash product $X \otimes_{\mathcal{E}} k$ inherits an R-module structure from k.

Definition 1.7. An R-module M is *effectively constructible* from k if the action map

$$\varphi\colon \operatorname{Hom}_R(k, M) \otimes_{\mathcal{E}} k \longrightarrow M$$

is an equivalence.

Notice that since $\operatorname{Hom}_R(k, M)$ is an $\mathcal{E}$-module, it is $\mathcal{E}$-cellular as an $\mathcal{E}$-module. Thus it is built from $\mathcal{E}$ as an $\mathcal{E}$-module, so $\operatorname{Hom}_R(k, M) \otimes_{\mathcal{E}} k$ is built from k as an R-module. Hence the domain of φ is always k-cellular. It follows that

$$\begin{aligned} \varphi \text{ is a } k\text{-equivalence} &\iff \varphi \text{ is a } k\text{-cellular approximation} \\ &\iff \operatorname{Cell}_k M \text{ is effectively constructible.} \end{aligned}$$

Theorem 1.8. *Assume that k is a small R-module. Then the adjunction*

$$D(\mathcal{E}^{\mathrm{op}}) \underset{\operatorname{Hom}_R(k,-)}{\overset{-\otimes_{\mathcal{E}} k}{\rightleftarrows}} \operatorname{Cell}_k D(R)$$

is an equivalence. The map φ is the counit of the adjunction, so it follows that all k-cellular R-modules are effectively constructible.

Proof. To construct the adunction using the language of enriched functors, start with the enriched adjunction

$$\operatorname{Hom}_S(X \otimes_S k, M) \cong \operatorname{Hom}_S(X, \operatorname{Hom}_S(k, M))$$

coming from the fact that Mod_S is closed symmetric monoidal under $- \otimes_S -$ and $\operatorname{Hom}_S(-,-)$. If we apply $\operatorname{Hom}_S(-, M)$ to the coequalizer description of the smash product over $\mathcal{E}$

$$X \otimes_S \mathcal{E} \otimes_S k \rightrightarrows X \otimes_S k \longrightarrow X \otimes_{\mathcal{E}} k,$$

and use the adjunction, we get the equalizer description of $\operatorname{Hom}_{\mathcal{E}^{\mathrm{op}}}$

$$\operatorname{Hom}_S(X \otimes \mathcal{E}, \operatorname{Hom}(k, M)) \leftleftarrows \operatorname{Hom}_S(X, \operatorname{Hom}(k, M)) \longleftarrow \operatorname{Hom}_{\mathcal{E}^{\mathrm{op}}}(X, \operatorname{Hom}_S(k, M))$$

Similarly, the equalizer description of $\operatorname{Hom}_R(-,-)$ transforms under the adjunction, so that we have an enriched adjunction

$$\operatorname{Hom}_R(X \otimes_{\mathcal{E}} k, M) \cong \operatorname{Hom}_{\mathcal{E}^{\mathrm{op}}}(X, \operatorname{Hom}_R(k, M)).$$

After passage to derived functors and homotopy categories, we aim to show that the adjunction induces an equivalence between $\mathcal{E}^{\mathrm{op}}$-modules and the full subcategory of k-cellular R-modules. The morphism $\operatorname{Hom}_R(k, \varphi)$ fits into the following commutative diagram, where the equivalences are given by the dualizability of the small R-module k.

$$\begin{array}{ccc}
\operatorname{Hom}_R(k, R) \otimes_R \operatorname{Hom}_R(k, M) \otimes_{\mathcal{E}} k & \xrightarrow{\simeq} & \operatorname{Hom}_R(k, \operatorname{Hom}_R(k, M) \otimes_{\mathcal{E}} k) \\
\text{swap} \downarrow \cong & & \\
\operatorname{Hom}_R(k, M) \otimes_{\mathcal{E}} \operatorname{Hom}_R(k, R) \otimes_R k & & \\
\downarrow \simeq & & \downarrow \operatorname{Hom}_R(k,\varphi) \\
\operatorname{Hom}_R(k, M) \otimes_{\mathcal{E}} \operatorname{Hom}_R(k, k) & & \\
\text{can} \downarrow \cong & & \\
\operatorname{Hom}_R(k, M) & \swarrow &
\end{array}$$

Thus φ is a k-equivalence of k-cellular R-modules, hence an equivalence. The unit η of the adjunction fits into the commutative diagram

$$\begin{array}{ccc} X \otimes_{\mathcal{E}} \mathrm{Hom}_R(k, R) \otimes_R k & \xrightarrow{\simeq} & X \otimes_{\mathcal{E}} \mathrm{Hom}_R(k, k) \\ \mathrm{swap} \downarrow \cong & & \mathrm{can} \downarrow \cong \\ \mathrm{Hom}_R(k, R) \otimes_R X \otimes_{\mathcal{E}} k & & X \\ \downarrow \simeq & & \\ \mathrm{Hom}_R(k, X \otimes_{\mathcal{E}} k) & \xleftarrow{\eta} & \end{array}$$

where the equivalences use the dualizability of k. Thus η is an equivalence as well. □

2. DC-completeness

We fix an S-algebra k (often commutative, but we need not assume so yet). Let $R \longrightarrow k$ be a map of S-algebras. In most of our examples, R is an augmented k-algebra, i.e. a k-algebra equipped with a map of k-algebras $R \longrightarrow k$. We make the following definitions:

- $R \longrightarrow k$ is *small* if k is finitely built from R as an R-module (recall Lemma 1.2).
- $R \longrightarrow k$ is *cosmall* if R is finitely built from k as an R-module.
- $R \longrightarrow k$ is *proxy-small* if there exists an R-module K such that
  - K is finitely built from R,
  - K is finitely built from k, and
  - k is built from K

  in the category of R-modules. We then call K a *Koszul complex* for k over R.

Remark 2.1. small $\Longrightarrow$ proxy-small ($K = k$), and
cosmall $\Longrightarrow$ proxy-small ($K = R$).

Example 2.2. Let R be a commutative Noetherian discrete ring, let I be the ideal of R generated by $\alpha_1, \ldots, \alpha_n$, and let $k = R/I$ be the quotient ring. Assume that k is a regular ring(= every finitely generated k-module admits a finite length resolution by finitely generated projective k-modules, i.e. is finitely built). We recall the classical Koszul complex and show that it witnesses the fact that $R \longrightarrow k$ is proxy-small.

Let $K(\alpha_i)$ be the two term chain complex

$$K(\alpha_i)\colon R \xrightarrow{\alpha_i} R$$

with R in degrees 0 and 1, respectively. Let $K = K(\alpha_1) \otimes_R \cdots \otimes_R K(\alpha_n)$ be the tensor product of chain complexes. In degree p, the chain complex K is the p-fold alternating product $K_p = \bigwedge^p R^n$ and, in terms of the usual basis for alternating products, the differential is given by

$$d(e_{i_0} \wedge \cdots \wedge e_{i_p}) = \sum_{k=0}^{p} (-1)^k \alpha_{i_k} e_{i_0} \wedge \cdots \wedge \widehat{e_{i_k}} \wedge \cdots \wedge e_{i_p}.$$

Appropriate modifications should be made in characteristic two. From this description it is clear that K is finitely built from R.

If $\alpha_1, \ldots, \alpha_n$ is a regular sequence, then K is a resolution of k. This is not true in general, but we will show that K is finitely built from k. First, I claim that for any n-tuple $r = (r_1, \ldots, r_n)$, multiplication by $v = \sum r_i \alpha_i$ is nullhomotopic. To construct an explicit chain nullhomotopy, define $H\colon K_* \longrightarrow K_{*+1}$ by $H(x) = r \wedge x$, where we consider $r \in R^n \cong K_1$ to be a degree one element of the exterior algebra on R^n. By the Leibniz rule,

$$dH(x) = dr \wedge x - r \wedge dx = \sum r_i \alpha_i x - H(dx).$$

Thus $dHx + Hdx = vx$, which proves the claim. It follows that in each degree, the homology H_iK is a finitely generate $k = R/I$-module, hence tiniely builtt from k. We now use the short exact sequences coming from the Postnikov tower

$$0 \longrightarrow H_iK \longrightarrow K\langle -\infty, i\rangle \longrightarrow K\langle -\infty, i-1\rangle \longrightarrow 0$$

and induction to see that $K = K\langle -\infty, n\rangle$ is finitely built from k.

To verify that K is a Koszul complex for k, it remains to see that k is built from K. Every R-module is built from R over R. In particular, $k = R/I$ is built from R. Applying $K \otimes_R (-)$ to the buliding procedure, we see that $K \otimes_R k$ is built from K over R. But after tensoring with $k = R/I$, the differential on K is zero and the complex splits as a finite sum of shifted copies of k:

$$K \otimes_R k \simeq \oplus_{\text{finite}} \Sigma^t k.$$

Using triangles to shed off copies of k, we conclude that k is built from K.

We will now introduce a duality functor which can be thought of as the linear dual of Koszul/bar duality. Recall that we have fixed a map of S-algebras $R \longrightarrow k$. Let $\mathcal{E} = \mathrm{End}_R(k)$ be the (derived) endomorphism algebra of k as an R-module. The association $R \mapsto \mathcal{E}$ defines a contravariant endofunctor when restricted to the category of augmented k-algebras. The unit of R induces the augmentation $\mathrm{End}_R(k) \longrightarrow \mathrm{End}_k(k) = k$ of the endomorphism ring. Applying the same procedure again gives the double centralizer $\widehat{R} = \mathrm{End}_{\mathcal{E}}(k)$ of R.

Definition 2.3. We say that (R, k) is *DC-complete* if the map induced from the R-module action on k

$$R \longrightarrow \mathrm{End}_{\mathcal{E}}(k) = \widehat{R}$$

is an equivalence.

Remark 2.4. If R is DC-complete, then it follows that $\mathcal{E}$ is also DC-complete.

Notice that the endomorphism ring of R is naturally isomorphic to the k-linear dual of $k \otimes_R k$:

$$\mathcal{E} = \mathrm{Hom}_R(k, k) \cong \mathrm{Hom}_R(k, \mathrm{Hom}_k(k, k)) \cong \mathrm{Hom}(k \otimes_R k, k).$$

The (derived) smash product $k \otimes_R k$ is canonically realized using the two-sided bar construction $B(k, R, k)$ on R. Thus the functor $R \longmapsto \mathcal{E}$ is the contravariant dual of the bar duality functor $R \longmapsto B(k, R, k)$. This functor is also popular in other contexts. For example making the same definitions for operads instead of augmented k-algebras gives the notion of Koszul duality of operads.

Example 2.5. Let R be a commutative Noetherian ring, let I be an ideal of R with quotient $k = R/I$ a regular ring. We will now see that there is a natural equivalence $\widehat{R} \simeq \lim_s R/I^s$ between the double centralizer of $R \longrightarrow k$ and the I-adic completion of R. Hence R is DC-complete if and only if R is I-adically complete.

Let $\mathcal{C}$ be the class of R-modules X for which the natural map

$$X \longrightarrow \mathrm{Hom}_{\mathcal{E}}(\mathrm{Hom}_R(X, k), k)$$

is an equivalence. Certainly $k \in \mathcal{C}$. Since $\mathcal{C}$ is closed under triangles, retracts, and equivalences, we see that

$$\{R\text{-modules finitely built from } k\} \subset \mathcal{C}.$$

Since R is Noetherian, I^s is a finitely generated R-module. Thus $I^s/I^{s+1} \cong I^s \otimes_R R/I$ is a finitely generated k-module, hence finitely built from k by regularity. Using the short exact sequence

$$0 \longrightarrow I^s/I^{s+1} \longrightarrow R/I^{s+1} \longrightarrow R/I^s \longrightarrow 0$$

and induction, we conclude that R/I^{s+1} is finitely built from k, and so $R/I^{s+1} \in \mathcal{C}$. We will now apply a theorem of Grothendieck on local cohomology:

$$\operatorname*{colim}_s \operatorname{Ext}^i_R(R/I^s, k) \cong \begin{cases} k & i = 0 \\ 0 & i \neq 0 \end{cases}$$

In our language, we can write this as:

$$\operatorname*{hocolim}_s \operatorname{Hom}_R(R/I^s, k) \simeq k.$$

This equivalence and the fact that $R/I^s \in \mathcal{C}$ give the desired chain of natural equivalences:

$$\begin{aligned} \widehat{R} = \operatorname{Hom}_{\mathcal{E}}(k,k) &\simeq \operatorname{Hom}_{\mathcal{E}}(\operatorname*{hocolim}_s \operatorname{Hom}_R(R/I^s, k), k) \\ &\cong \operatorname*{holim}_s \operatorname{Hom}_{\mathcal{E}}(\operatorname{Hom}_R(R/I^s, k), k) \\ &\simeq \operatorname*{holim}_s R/I^s. \end{aligned}$$

Example 2.6. As Justin explained during the seminar, it is instructive to work out the case $R = \mathbf{Z} \longrightarrow \mathbf{Z}/p = k$ of the previous example by hand. The derived endomorphism algebra

$$\mathcal{E} = \operatorname{Hom}_{\mathbf{Z}}(\mathbf{Z}/p, \mathbf{Z}/p)$$

can be constructed by applying $\operatorname{Hom}_{\mathbf{Z}}(-, \mathbf{Z}/p)$ to the standard resolution

$$0 \longrightarrow \mathbf{Z} \xrightarrow{p} \mathbf{Z} \longrightarrow \mathbf{Z}/p \longrightarrow 0$$

of $\mathbf{Z}/p$. The result in homology is the exterior algebra $\Lambda = E(\mathbf{Z}/p)$ on a copy of $\mathbf{Z}/p$ in degree one. There is a spectral sequence

$$\operatorname{Ext}_{H_*\mathcal{E}}(\mathbf{Z}/p, \mathbf{Z}/p) \Longrightarrow H_* \operatorname{Ext}_{\mathcal{E}}(\mathbf{Z}/p, \mathbf{Z}/p)$$

whose target is the homology of the double centralizer $\operatorname{End}_{\mathcal{E}}(\mathbf{Z}/p)$. The tensor product $\Lambda \otimes \Gamma(\mathbf{Z}/p)$ of the exterior algebra with a divided power algebra on $\mathbf{Z}/p$ in degree -1 is a free resolution of $\mathbf{Z}/p$ as a $H_*\mathcal{E} = \Lambda$-module. Hence the E_2 page of the spectral sequence is isomorphic to a polynomial algebra on a copy of $\mathbf{Z}/p$ in homological degree 1 and internal degree 0:

$$\operatorname{Hom}_\Lambda(\Lambda \otimes \Gamma(\mathbf{Z}/p), \mathbf{Z}/p) \cong \operatorname{Hom}(\Gamma(\mathbf{Z}/p), \mathbf{Z}/p) \cong P(\mathbf{Z}/p).$$

For degree reasons, the spectral sequence collapses, and so the associated graded of $H_* \operatorname{Ext}_{\mathcal{E}}(\mathbf{Z}/p, \mathbf{Z}/p)$ is a copy of $\mathbf{Z}/p$ in each non-negative degree. As we know from the previous example, all of the extensions are non-trivial and the double centralizer is the p-adic integers $\mathbf{Z}_p^\wedge$.

Example 2.7. Let $R \longrightarrow k$ be the map $S_p \longrightarrow \mathbf{Z}/p$ from the p-complete sphere spectrum to the Eilenberg-MacLane spectrum of $\mathbf{Z}/p$ given by the first Postnikov section composed with reduction modulo p. The homotopy of the derived endormorphism algebra $\mathcal{E} = \operatorname{End}_{S_p}(\mathbf{Z}/p)$ is the Steenrod algebra $\pi_*\mathcal{E} = \mathcal{A}$. The double centralizer is the abutment of the Adams spectral sequence constructed by taking an Adams resolution of $\mathbf{Z}/p$

$$\operatorname{Hom}_{\mathcal{A}}(\mathbf{Z}/p, \mathbf{Z}/p) \Longrightarrow \pi_* \operatorname{Hom}_{\mathcal{E}}(\mathbf{Z}/p, \mathbf{Z}/p).$$

The statement that the Adams spectral sequence converges to the homotopy of S_p can then be interpreted as the fact that $S_p \longrightarrow \mathbf{Z}/p$ is DC-complete. Note that the failure of the Adams spectral sequence to converge to the homotopy of the global sphere S means that $S \longrightarrow \mathbf{Z}/p$ is not DC-complete.

Example 2.8. Let k be a commutative S-algebra. Let X be a pointed connected topological space. Let $R = C_*(\Omega X) = \Sigma^\infty \Omega X_+ \otimes k$ be the k-chains on the based loopspace ΩX, considered as a k-algebra by choosing a model for ΩX as a monoid. Notice that $C_*(\Omega X)$ is augmented over

k via the collapse map $\Omega X \longrightarrow *$. Let $C^*(X) = F_S(\Sigma^\infty X_+, k)$ be the k-chains on X. Notice that $C^*(X)$ is augmented over k via the inclusion of the basepoint of X.

Lemma 2.9. *There is a natural equivalence $\mathcal{E} = \mathrm{End}_{C_*(\Omega X)}(k) \simeq C^*(X)$ of augmented k-alegbras.*

Proof. The functor

$$\Sigma^\infty(-)_+ \otimes k \colon (\mathrm{Top}, \times) \longrightarrow (\mathrm{Mod}_k, \otimes_k)$$

is strong symmetric monoidal, so there is a natural isomorphism of bar constructions

$$\Sigma^\infty(B\Omega X)_+ \otimes k \cong B(k, C_*(\Omega X), k)$$

The left side is equivalent to the k-chains $C_*(X)$ on X and the right side is an explicit model for the derived smash product $k \otimes_{C_*(\Omega X)} k$. Taking the k-linear dual, we find that

$$C^*(X) = \mathrm{Hom}_S(C_*(X), k) \simeq \mathrm{Hom}_S(k \otimes_{C_*(\Omega X)} k, k) \cong \mathrm{Hom}_{C_*(\Omega X)}(k, k) = \mathcal{E}.$$

□

It follows that we can compute the double centralizer of $R = C_*(\Omega X)$ using cochains $C^*(X)$ instead of $\mathcal{E}$ itself. We will now introduce conditions under which $C_*(\Omega X)$ is DC-complete.

Definition 2.10. We say that (X, k) is of Eilenberg-Moore type if

- k is a field,
- $H_*(X; k)$ is finite type, and
- either $\pi_1 X = 0$ or k is characteristic p and $\pi_1 X$ is a finite p-group.

Proposition 2.11. *Assume that (X, k) is of Eilenberg-Moore type. Then $C_*(\Omega X) \simeq \mathrm{End}_{C^*(X)}(k)$. Consequently, both $C_*(\Omega X)$ and $C^*(X)$ are DC-complete.*

Proof. This can be interpreted in terms of the convergence of the Eilenberg-Moore spectral sequence. By taking the k-linear dual, it suffices to prove that the Eilenberg-Moore map

$$k \otimes_{C^*(X)} k \longrightarrow C^*(\Omega X)$$

defined in terms of the homotopy pullback diagram

$$\begin{array}{ccc} \Omega X & \longrightarrow & * \\ \downarrow & & \downarrow \\ * & \longrightarrow & X \end{array}$$

is an equivalence. We will prove more generally that given a homotopy pullback diagram

$$\begin{array}{ccc} P & \longrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

with (X, k) of Eilenberg-Moore type, the induced map

$$C^*(Y) \otimes_{C^*(X)} C^*(Z) \longrightarrow C^*(P)$$

is an equivalence. □

Returning to the general framework of an augmented k-algebra R, we will consider how the hypothesis that R is DC-complete relates smallness conditions on R and $\mathcal{E}$.

Proposition 2.12. *Assume that (R, k) is DC-complete.*

(i) $R \longrightarrow k$ is small if and only if $\mathcal{E} \longrightarrow k$ is cosmall.
(ii) $R \longrightarrow k$ is proxy-small if and only if $\mathcal{E} \longrightarrow k$ is proxy-small.

Proof. (i) Assume that k is finitely built from R as an R-module. Applying $\mathrm{Hom}_R(k,-)$, we see that $\mathcal{E} = \mathrm{Hom}_R(k,k)$ is finitely built from $k = \mathrm{Hom}_R(R,k)$ as an $\mathcal{E}$-module. Conversely, if $\mathcal{E}$ is finitely built from k, we apply $\mathrm{Hom}_{\mathcal{E}}(-,k)$ and see that $k = \mathrm{Hom}_{\mathcal{E}}(\mathcal{E},k)$ is finitely built from $R \simeq \mathrm{Hom}_{\mathcal{E}}(k,k)$ as an R-module.

(ii) Suppose that $R \longrightarrow k$ is proxy-small with Koszul complex K. Let $L = \mathrm{Hom}_R(K,k)$. We will show that L is a Koszul complex for k over $\mathcal{E}$. By similar reasoning as in (i),

$$K \text{ is finitely built from } R \Longrightarrow L \text{ is finitely built from } k$$
$$K \text{ is finitely built from } k \Longrightarrow L \text{ is finitely built from } \mathcal{E}$$

Using a result from last lecture we see that the composition map

$$\varphi\colon\ \mathrm{Hom}_R(K,k) \otimes_{\mathrm{Hom}_R(K,K)} K \longrightarrow k$$

is an equivalence of $\mathcal{E}$-modules because K is a small R-module. By taking any cellular approximation, K is built from $\mathcal{E}_K = \mathrm{End}_R(K)$ as an $\mathcal{E}_K$-module. Thus $k \simeq L \otimes_{\mathcal{E}_K} K$ is built from $L = L \otimes_{\mathcal{E}_K} \mathcal{E}_K$ as an $\mathcal{E}$-module. Therefore L is a Koszul complex and $\mathcal{E} \longrightarrow k$ is proxy-small. The converse follows from the same argument and the equivalence $R \simeq \mathrm{End}_{\mathcal{E}}(k)$ as in (i). □

Example 2.13. Assume that (X,k) is of Eilenberg-Moore type, so that $R = C_*(\Omega X)$ and $\mathcal{E} \simeq C^*(X)$ are both DC-complete. We have logical equivalences:

$$C_*(\Omega X) \to k \text{ small} \overset{\text{Prop}}{\Longleftrightarrow} C^*(X) \to k \text{ cosmall} \overset{(A)}{\Longleftrightarrow} H_*(X;k) \text{ finite dimensional}$$
$$C^*(X) \to k \text{ small} \overset{\text{Prop}}{\Longleftrightarrow} C_*(\Omega X) \to k \text{ cosmall} \overset{(B)}{\Longleftrightarrow} H_*(\Omega X;k) \text{ finite dimensional}$$

In fact, the equivalence (A) holds only under the hypothesis that k is a field. The implication ($\Longrightarrow$) is obvious, and to prove the reverse implication we use the coPostnikov tower

$$\cdots \longrightarrow \mathcal{E}\langle n,\infty\rangle \longrightarrow \mathcal{E}\langle n+1,\infty\rangle \longrightarrow \cdots$$

which furnishes a cofiber sequence of spectra

$$\mathcal{E}\langle n,\infty\rangle \longrightarrow \mathcal{E}\langle n+1,\infty\rangle \longrightarrow K(\pi_n\mathcal{E}, n).$$

(The existence of the coPostnikov tower uses that $\mathcal{E}$ is coconnective with $\pi_0\mathcal{E} = H^0(X;k) = k$ a field; this is worked out carefully by hand in Dwyer-Greenlees-Iyengar Prop 3.3.) The Eilenberg-MacLane spectrum is finitely built from K because $\pi_n\mathcal{E} = H^{-n}(X;k)$ is finite dimensional. Since the total cohomology ring $H^*(X;k)$ is finite dimensional, we may induct along such cofiber sequences and conclude that $\mathcal{E}$ is finitely built from K.

The implication (B) is similar, but requires the full assumption that (X,k) is Eilenberg-Moore type. Again, the implication ($\Longrightarrow$) is obvious. For the reverse implication, we use the Postnikov tower (whose construction only uses that $R = C_*(\Omega X)$ is connective):

$$\cdots \longrightarrow R\langle -\infty, n\rangle \longrightarrow R\langle -\infty, n-1\rangle \longrightarrow \cdots$$

with associated fiber sequences

$$K(\pi_n R, n) \longrightarrow R\langle -\infty, n\rangle \longrightarrow R\langle -\infty, n-1\rangle.$$

Under our hypotheses, there is a finite filtration of $\pi_n R$ by $\pi_0 R = k[\pi_1 X]$-modules

The Eilenberg-MacLane spectrum $K(\pi_n R, n)$ is finitely built from k as an R-module.

3. Matlis lifts and Matlis duality

As before, let $R \longrightarrow k$ be a map of S-algebras and set $\mathcal{E} = \mathrm{End}_R(k)$.

Definition 3.1. Let N be a k-module. We say that an R-module $\mathcal{I} = \mathcal{I}_N$ is a *Matlis lift* of N if

(1) there is an equivalence of k-modules $\mathrm{Hom}_R(k, \mathcal{I}) \simeq N$, and

(2) $\mathcal{I}$ is effectively constructible from k, i.e. the natural map
$$\mathrm{eval}\colon \mathrm{Hom}_R(k, \mathcal{I}) \otimes_{\mathcal{E}} k \longrightarrow \mathcal{I}$$
is an equivalence.

Given $\mathcal{I}$ satisfying condition (1), a k-cellular approximation $\mathrm{Cell}_k\mathcal{I}$ also satisfies (1). So there is no loss in generality in assuming that $\mathcal{I}$ is k-cellular. Instead of requiring this, we make impose the stronger condition (2) since it will help enumerate Matlis lifts below.

Remark 3.2. The functor $\mathrm{Hom}_R(k, -)$ is the coinduction functor from R-modules to k-modules, so from condition (1) we get a canonical equivalence
$$\mathrm{Hom}_R(X, \mathcal{I}) \simeq \mathrm{Hom}_k(X, N)$$
for any R-module X. Notice that $\mathcal{I}$ is a lift of N to an R-module, but not in the obvious way along the map $R \longrightarrow k$.

Remark 3.3. When $N = k$, then
$$\mathrm{Hom}_R(X, \mathcal{I}) \simeq \mathrm{Hom}_k(X, k)$$
is a lift of the k-linear duality functor to the setting of R-modules.

Definition 3.4. A right $\mathcal{E}$-module $\widetilde{N}$ is of *Matlis type* if the coevaluation map
$$\mathrm{coeval}\colon \widetilde{N} \cong \widetilde{N} \otimes_{\mathcal{E}} \mathrm{Hom}_R(k,k) \longrightarrow \mathrm{Hom}_R(k, \widetilde{N} \otimes_{\mathcal{E}} k)$$
is an equivalence. If the underlying left k-module of $\widetilde{N}$ along the right multiplication map $k^{\mathrm{op}} \longrightarrow \mathcal{E}$ is equivalent to N, then we call $\widetilde{N}$ an *$\mathcal{E}$-lift* of N.

Proposition 3.5. *There is a bijective correspondence of equivalence classes:*
$$\pi_0\{\textit{Matlis lifts of } N\} \cong \pi_0\{\mathcal{E}\textit{-lifts of } N \textit{ of Matlis type }\}$$
$$\mathcal{I} \longmapsto \mathrm{Hom}_R(k, \mathcal{I})$$
$$\widetilde{N} \otimes_{\mathcal{E}} k \longleftarrow \widetilde{N}$$

We will be most interested in the case $N = k$. To find a Matlis lift $\mathcal{I}$ of k, it suffices to

(1) define a compatible $\mathcal{E}^{\mathrm{op}}$-module structure on k, i.e. an $\mathcal{E}$-lift $\widetilde{k}$ of k, and then

(2) prove that $\widetilde{k}$ is of Matlis type.

The resulting Matlis lift is then of the form $\mathcal{I} = \widetilde{k} \otimes_{\mathcal{E}} k$. In this situation, we make the following definition.

Definition 3.6. The *Pontriagin dual* or *Matlis dual* of an R-module M (with respect to $\mathcal{I} = \mathcal{I}_k$) is defined to be $\mathrm{Hom}_R(M, \mathcal{I})$. Note that without further structure on R, for example commutativity, this need not be an R-module!

We will now provide a series of propositions which enable us to recognize $\mathcal{E}$-lifts of Matlis type and thus Matlis lifts.

Proposition A. Suppose that $R \longrightarrow k$ is small. Then any $\mathcal{E}$-lift $\widetilde{N}$ of N is of Matlis type. Consequently, $\mathcal{I} = \widetilde{N} \otimes_{\mathcal{E}} k$ is a Matlis lift of N.

Proof. By Theorem 1.8, the unit

$$\text{coeval}\colon \widetilde{N} \longrightarrow \mathrm{Hom}_R(k, \widetilde{N} \otimes_{\mathcal{E}} k)$$

of the adjunction between $\mathcal{E}^{\mathrm{op}}$-modules and k-cellular R-modules is an equivalence. □

Proposition B. Let M be an R-module. Then the $\mathcal{E}^{\mathrm{op}}$-module $\mathrm{Hom}_R(k, M)$ is of Matlis type if and only if

$$\text{eval}\colon \mathrm{Hom}_R(k, M) \otimes_{\mathcal{E}} k \longrightarrow M$$

is a k-cellular approximation.

Notice that the domain of the map eval is the associated Matlis lift $\mathcal{I}$. Also notice that the second condition is equivalent to the statement that $\mathrm{Cell}_k M$ is effectively constructible.

Remark 3.7. Suppose that R is an augmented k-algebra with k commutative, and consider the $\mathcal{E}$-lift $\widetilde{k}$ of k given by the augmentation. In some examples, we will already know that the $\mathcal{E}$-lift k is of Matlis type. The adjunction

$$\mathrm{Hom}_R(k, \mathrm{Hom}_k(R, k)) \cong \mathrm{Hom}_k(k \otimes_R R, k) \cong k$$

then shows that $\mathrm{Hom}_R(k, \mathrm{Hom}_k(R, k))$ is Matlis type. Applying Proposition B with $M = \mathrm{Hom}_k(R, k)$ gives a k-cellular approximation

$$\mathcal{I} = \widetilde{k} \otimes_{\mathcal{E}} k \simeq \mathrm{Hom}_R(k, \mathrm{Hom}_k(R, k)) \otimes_{\mathcal{E}} k \longrightarrow \mathrm{Hom}_k(R, k),$$

so that $\mathcal{I} = \mathrm{Cell}_k \mathrm{Hom}_k(R, k)$.

Corollary 3.8. *If $R \longrightarrow k$ is proxy-small, then $\widetilde{N}$ is of Matlis type if and only if there exists an R-module M such that $\widetilde{N} \simeq \mathrm{Hom}_R(k, M)$ as $\mathcal{E}^{\mathrm{op}}$-modules.*

Proof. If $\widetilde{N}$ is of Matlis type, then $M = \widetilde{N} \otimes_{\mathcal{E}} k$ has the desired property. The converse follows from the generalization of Theorem 1.8 to the proxy-small case (see DGI Theorem 4.10). □

Definition 3.9. An R-module X is of upward (finite) type if there exists an integer n such that X can be built from 0 by attaching (finitely many) copies of $\Sigma^i R$ for each $i \geq n$. An R-module X is of downward (finite) type if there exists an integer n such that X can be built from 0 by attaching (finitely many) copies of $\Sigma^i R$ for each $i \leq n$.

The following lemma is useful to recognize when these conditions hold.

Lemma 3.10. *(see DGI Prop 3.13 and Prop 3.14)*

(i) *If R is connective and X is bounded below, then X is upward type. If $\pi_0 R$ is Noetherian and each $\pi_i R$, $\pi_i X$ is a finitely generated $\pi_0 R$-module, then X is upward finite type.*

(ii) *If R is coconnective, $\pi_0 R$ is a field and X is bounded above, then X is downwarad type. If in addition $\pi_{-1} R = 0$ and $\pi_i R$, $\pi_i X$ are each finitely generated over $\pi_0 R$, then X is of downward finite type.*

Proof. For (i), we inductively construct maps of R-modules $X_n \longrightarrow X$ that is a π_i-iso for $i < n$ and surjective on π_n. For the inductive step, choose maps $\amalg \Sigma^n R \longrightarrow X_n$ generating the kernel of $\pi_n X_n \longrightarrow \pi_n X$. Let X'_n be the cofiber of this map. There is an induced map $X'_n \longrightarrow X$ which is now a π_i-iso for $i \leq n$. Next, choose maps $Y = \amalg \Sigma^{n+1} R \longrightarrow X$ generating $\pi_{n+1} X$. Then the induced map $Y \amalg X'_n \longrightarrow X$ has the required properties. Under the finiteness hypotheses, the coproducts can all be chosen to be finite.

The proof of (ii) is in some sense Eckmann-Hilton dual to the proof just given. It proceeds using a coPostnikov tower for X, which requires the assumptions on R. □

Again, we consider an $\mathcal{E}$-lift $\widetilde{N}$ of a k-module N and ask if it is of Matlis type.

Proposition C. Suppose that k and $\widetilde{N}$ are bounded above, that k is upward finite type as an R-module and $\widetilde{N}$ is downward type as an $\mathcal{E}^{\mathrm{op}}$-module. Then $\widetilde{N}$ is of Matlis type.

Proposition D. Suppose that k and $\widetilde{N}$ are bounded below, that k is downward finite type as an R-module and $\widetilde{N}$ is upward type as an $\mathcal{E}^{\mathrm{op}}$-module. Then $\widetilde{N}$ is of Matlis type.

We will now discuss some examples of Matlis lifts and Matlis duality.

Example 3.11. Suppose that $R \longrightarrow k$ is the unit map $S \longrightarrow H\mathbf{F}_p$ (see also Example 2.7). Then $\mathcal{E} = \mathrm{Hom}_S(H\mathbf{F}_p, H\mathbf{F}_p) = H\mathcal{A}$ is the spectrum representing the mod p Steenrod algebra, i.e. $\pi_k\mathcal{E} = \mathcal{A}^{-k}$. The $\pi_0\mathcal{E} = \mathbf{F}_p$-module structure on $\pi_k H\mathbf{F}_p$ is unique, so there exists a unique $\mathcal{E}^{\mathrm{op}}$-module structure on $k = H\mathbf{F}_p$ (given by the augmentation of the Steenrod algebra). This is our $\mathcal{E}$-lift. The conditions in Lemma 3.10.(i) are satisfied, so $H\mathbf{F}_p$ is of upward finite type as an S-module. We can also apply Lemma 3.10.(ii) to $H\mathbf{F}_p$ as an $\mathcal{E}^{\mathrm{op}}$-module, so $H\mathbf{F}_p$ is of downard type as an $\mathcal{E}^{\mathrm{op}}$-module. Notice that $\pi_{-1}\mathcal{E} \neq 0$ because of the Bockstein operation, and that this is the only condition for $H\mathbf{F}_p$ to be of downard finite type that does not hold. We conclude by Proposition C that $H\mathbf{F}_p$ is of Matlis type, and so $\mathcal{I} = H\mathbf{F}_p \otimes_{\mathcal{E}} H\mathbf{F}_p$ is a Matlis lift of $H\mathbf{F}_p$ (in fact, the unique Matlis lift).

We will now identify the resulting Matlis duality as the p-primary part of Brown-Comenetz duality. Recall that the Brown-Comenetz dual of an S-module E is the S-module JE satisfying the equation

$$JE^k(X) = \mathrm{Hom}(E_*(X), \mathbf{Q}/\mathbf{Z}).$$

In fact, we construct JE by observing that the right-hand side of this equation satisfies Brown representability. Let $J = JS$ be the Brown-Comenetz dual of the sphere spectrum. Then the homotopy groups of J

$$\pi_k J = J^{-k}(*) = \mathrm{Hom}(\pi_{-k}S, \mathbf{Q}/\mathbf{Z})$$

are the Pontrjagin duals of the satable stems (with the opposite degrees). Let J_p be the p-localization of J. Then the homotopy groups of J_p

$$\pi_k J_p = \mathrm{Hom}(\pi_{-k}S, \mathbf{Z}[1/p]/\mathbf{Z})$$

are the Pontrjagin duals of the p-primary stable stems. Observe that

$$J_p^{-k}(H\mathbf{F}_p) = \mathrm{Hom}(\pi_k H\mathbf{F}_p, \mathbf{Z}[1/p]/\mathbf{Z}) = \begin{cases} \mathbf{F}_p & k = 0 \\ 0 & \text{else} \end{cases}$$

Therefore $\mathrm{Hom}_S(H\mathbf{F}_p, J_p) \simeq H\mathbf{F}_p$, i.e. $H\mathbf{F}_p$ is Brown-Comenetz self-dual. It follows that the $\mathcal{E}^{\mathrm{op}}$-module $= Hom_S(H\mathbf{F}_p, J_p)$ is of Matlis type, so by Proposition B,

$$\mathrm{eval}\colon \mathrm{Hom}_S(H\mathbf{F}_p, J_p) \otimes_{\mathcal{E}} H\mathbf{F}_p \longrightarrow J_p$$

is a k-cellular approximation.

Now I claim that J_p is $H\mathbf{F}_p$-cellular, so that the map eval is in fact an equivalence. To see this, write J_p as the homotopy colimit of its connective covers

$$J_p = \operatorname*{hocolim}_n J_p\langle -n, \infty\rangle.$$

Each $J_p\langle -n, \infty\rangle$ has finitely many homotopy groups, each of which is a finite p-group, except for $\mathrm{Hom}(\mathbf{Z}, \mathbf{Z}[1/p]/\mathbf{Z}) = \mathbf{Z}[1/p]/\mathbf{Z}$ in degree 0. But all of these groups are built from $\mathbf{F}_p$ using short exact sequences (and the colimit $\mathbf{Z}[1/p]/\mathbf{Z} = \mathrm{colim}\, \mathbf{Z}/p^n$). We conclude that J_p is built from $H\mathbf{F}_p$.

We now have an equivalence

$$\mathcal{I} = H\mathbf{F}_p \otimes_{\mathcal{E}} H\mathbf{F}_p \simeq \mathrm{Hom}_S(H\mathbf{F}_p, J_p) \otimes_{\mathcal{E}} H\mathbf{F}_p \simeq J_p$$

and so Matlis duality is p-primary Brown-Comenetz duality.

Example 3.12. Let X be a pointed connected topological space. Let k be a field and suppose that (X, k) is of Eilenberg-Moore type. Let $R = C^*(X; k)$. Then $\mathcal{E} \simeq C_*(\Omega X; k)$ by DC-completeness (see Example 2.11). Then $\pi_0 R = H^0(X; k) = k$ and $\pi_i R$ is finite dimensional over k by the assumption that X is finite type. Similarly, $\pi_0 \mathcal{E} = H_0(\Omega X; k) = k[\pi_1 X]$ is Noetherian since (X, k) is of EM-type, and $\pi_i \mathcal{E}$ and k are finitely generated $k[\pi_1 X]$-modules. We apply Lemma 3.10 and conclude that k is upward finite type over $\mathcal{E}^{\mathrm{op}}$ and that k is downard finite type over R. Hence by Proposition D, $\mathcal{I} = k \otimes_{C_*(\Omega X)} k$ is a Matlis lift of k. Using the bar construction to model the derived smash product, we find that

$$\mathcal{I} \simeq C_X(B\Omega X; k) \simeq C_*(X; k) = \mathrm{Hom}_k(R, k).$$

The situation is as described in Remark 3.7, but $\mathcal{I}$ is in fact equivalent to $\mathrm{Hom}_k(R, k)$, not just a k-cellular approximation of it.

Before beginning the next example, let us record:

Lemma 3.13. *Suppose that X is a pointed connected finite complex, and that k is an S-algebra. Let $R = C_*(\Omega X; k)$ so that $\mathcal{E} \simeq C^*(X; k)$. Then k is small as an R-module and cosmall as an $\mathcal{E}$-module. If we merely assume that X is finite type, then k is of upward finite type over R.*

Proof. Let E be the total space of the universal principal ΩX-bundle over X. Then $M = C_*(E; k) \simeq k$ and the action of R on M agrees with the augmentation action on k. Let E_q be the inverse image in E of the q-skeleton of X, and let $M_q = C_*(E_q; k)$. The inverse image of a q-cell $e \subset X$ is a copy of $e \times \Omega X$ in E. Thus

$$M_q/M_{q-1} \simeq \bigoplus_{q\text{-cells in } X} \Sigma^q R.$$

Since X is finite dimensional, the filtration stabilizes and we see that $k \simeq M = M_{\dim X}$ is finitely built from R as an R-module. Applying $\mathrm{Hom}_R(-, k)$ to the construction process, we see that $\mathcal{E}$ is finitely built from k as an $\mathcal{E}$-module. If X is only finite type, then the filtration shows that k is of upward finite type as an R-module. □

Example 3.14. As in the previous example, assume that X is a pointed connected topological space of finite type and that k is a field. We will not assume that (X, k) is EM-type. Let $R = C_*(\Omega X; k)$. Then $\mathcal{E} \simeq C^*(X; k)$. The inclusion of the basepoint induces the augmentation $\mathcal{E} \longrightarrow k$, and this gives the unique $\mathcal{E}$-lift of k. By the lemma, we see that k is of upward finite type as a $C_*(\Omega X)$-module. By Lemma 3.10.(ii), k is of downward type as an $\mathcal{E}^{\mathrm{op}}$-module. By Proposition C, we conclude that $\mathcal{I} = k \otimes_{C^*(X)} k$ is the unique Matlis lift of k. Arguing as in Remark 3.7, we see that $\mathcal{I}$ is the k-cellular approximation of the cochains on the based loop spaces:

$$\mathcal{I} \simeq \mathrm{Cell}_k \mathrm{Hom}_k(R, k) = \mathrm{Cell}_k C^*(\Omega X; k)$$

This gives a conceptual interpretation of the target of the Eilenberg-Moore spectral sequence in cases where classical convergence does not hold.

Example 3.15. Let X be a pointed finite complex, let $k = S$, and let $R = C_*(\Omega X; k) = \Sigma^\infty_+ \Omega X$. Then $\mathcal{E} \simeq C^*(X; k) = DX$ is the Spanier-Whitehead dual of X. The ring spectrum R is Waldhausen's spherical group-ring of the based loopspace, and that the algebraic K-theory of R is the A-theory of X. By Lemma 3.13, $k = S$ is small as an R-module so by Proposition A, every $\mathcal{E}$-lift of S is of Matlis type. The augmentation $\mathcal{E} = DX \longrightarrow S$ gives the unique $\mathcal{E}$-lift of S and we conclude that $\mathcal{I} = S \otimes_{DX} S$ is the unique Matlis lift of S. As in Remark 3.7, we have

$$\mathcal{I} = S \otimes_{DX} S \simeq \mathrm{Hom}_S(\Sigma^\infty_+ \Omega X, S).$$

Suppose in addition that X is 1-connected. Realizing the derived smash product using the bar construction and applying $\mathrm{Hom}_S(-, S)$ in each simplical degree, we see that the double centralizer is the cobar construction on $\Sigma^\infty_+ X$, which is equivalent to $\Sigma^\infty_+ \Omega X$ by a theorem of Bousfield:

$$\mathrm{End}_{\mathcal{E}}(S) \cong \mathrm{Hom}(S \otimes_{DX} S, S) = \mathrm{Tot}\, C^*(X, \Sigma^\infty_+ X, S) \simeq \Sigma^\infty_+ \Omega X = R.$$

Therefore $(\Sigma^\infty_+ \Omega X, S)$ is DC-complete when X is 1-connected.