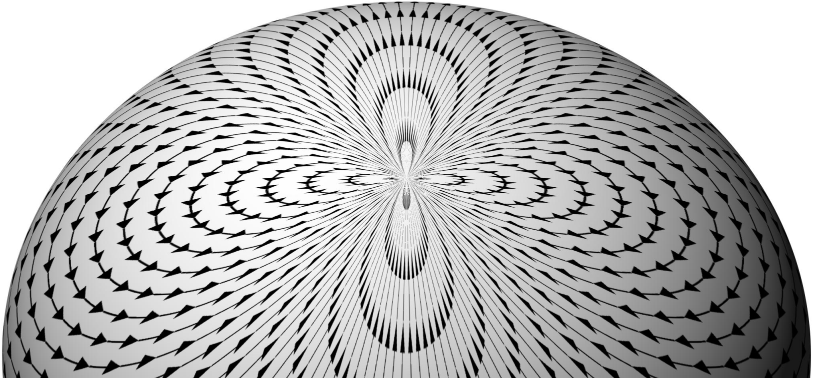
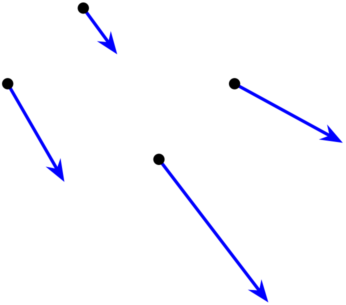
Vector fields on spheres

John Lind

February 28, 2019



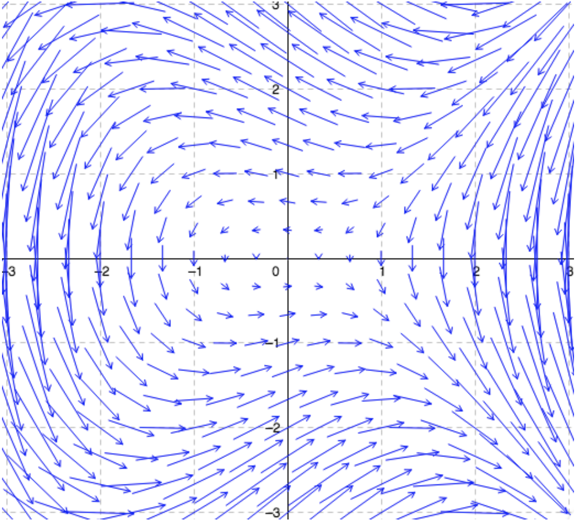
A *vector field* consists of a vector emanating from every point:

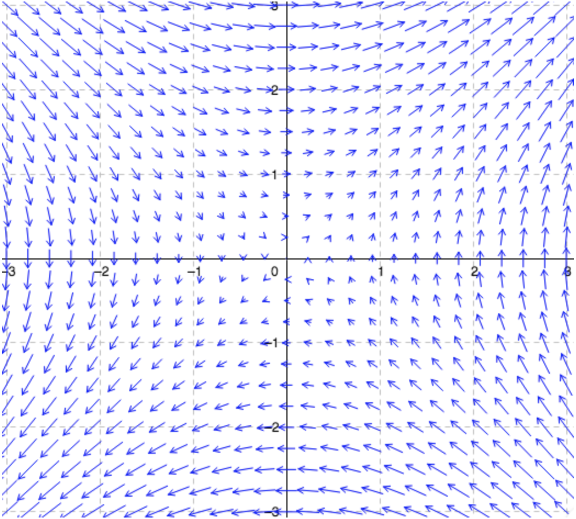


A vector field determines a *flow* through space in the direction that the vectors point.

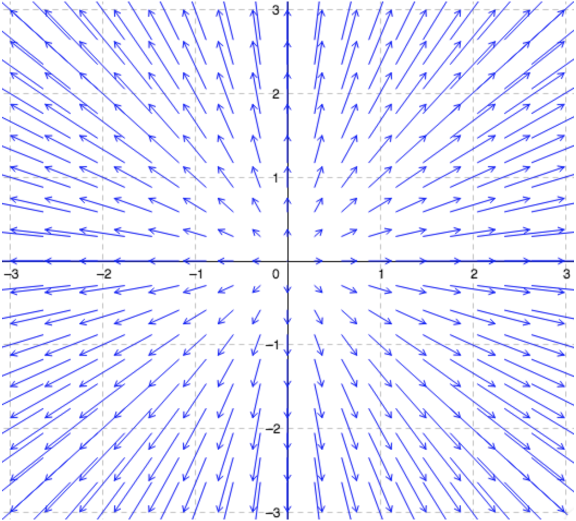
Technical disclaimer: the vectors should vary continuously as we move around.

A vector field in the plane:





$$F(x, y) = (y, x)$$



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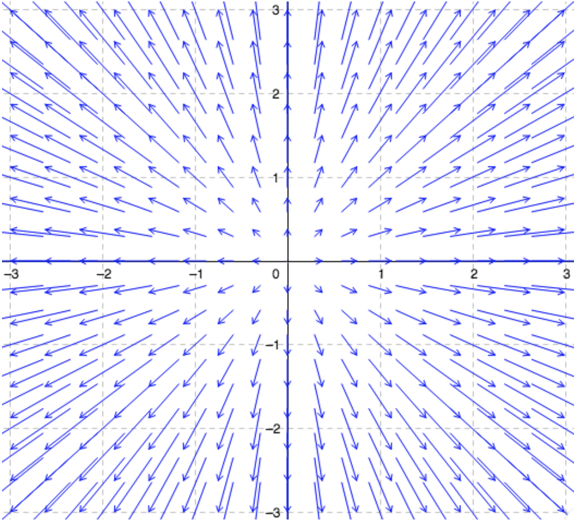
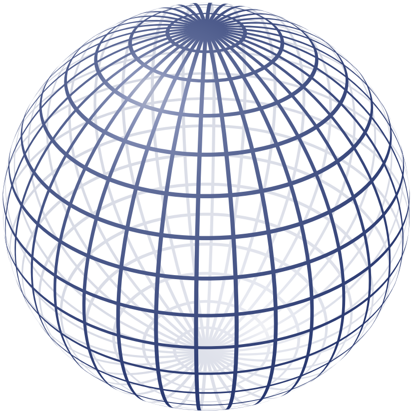


Q: Can you put this vector field on a sphere?

The *two-dimensional sphere* S^2 is the set of points in 3-space whose distance from the origin is one:

$$S^2 = \{(x, y, z) \in \mathbf{R}^3 \mid x^2 + y^2 + z^2 = 1\}$$

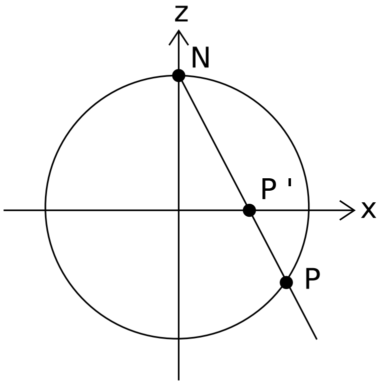


We can think of the two-dimensional sphere as the plane with an extra point ∞ "in all directions at once"

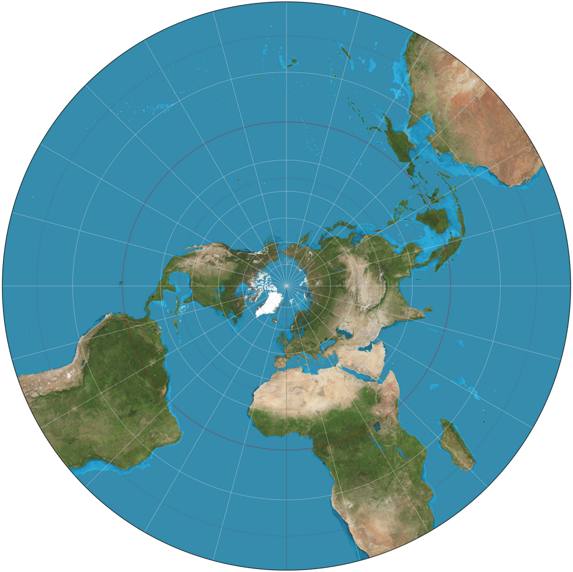
$$S^2 \cong \mathbf{R}^2 \cup \{\infty\}.$$

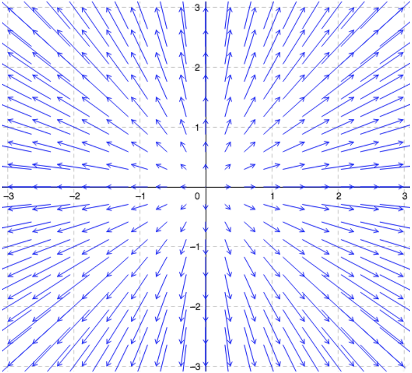
This is called *stereographic projection*. The formula is:

$$P = (x, y, z) \longmapsto \left(\frac{x}{1-z}, \frac{y}{1-z}\right) = P'.$$

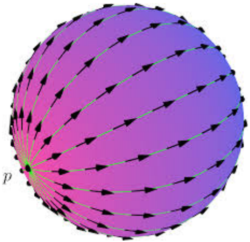


The stereographic projection of the earth:





When placed on the sphere, this vector field looks like:

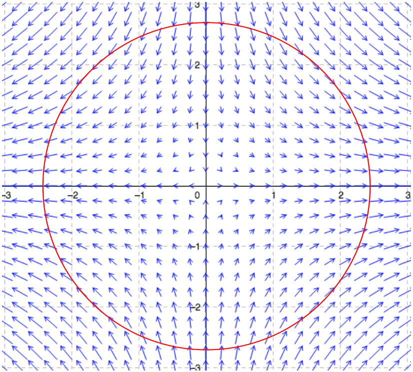


Q: can you find a *nonzero* vector field on the sphere?

Q: can you find a *nonzero* vector field on the sphere?

A point p is a *zero* of a vector field F if $F(p) = \vec{0}$. We want to find a vector field on S^2 without any zeroes.

The *index* of p is the number of times that the vector field make a full rotation (in $+$ or $-$ direction) as we circumnavigate p.



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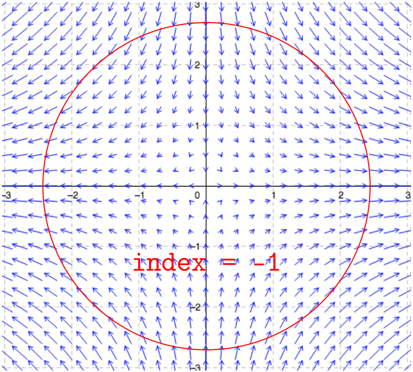

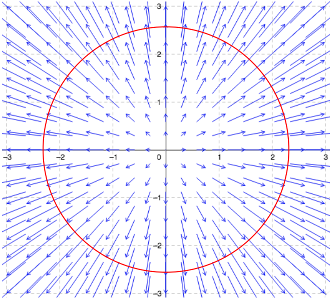
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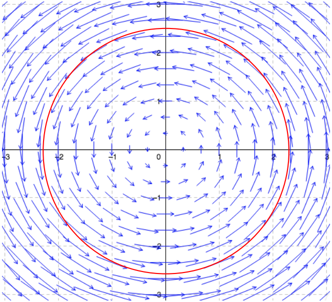
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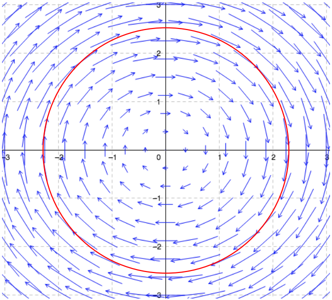
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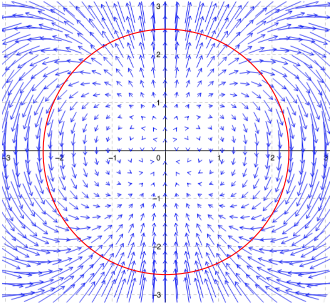
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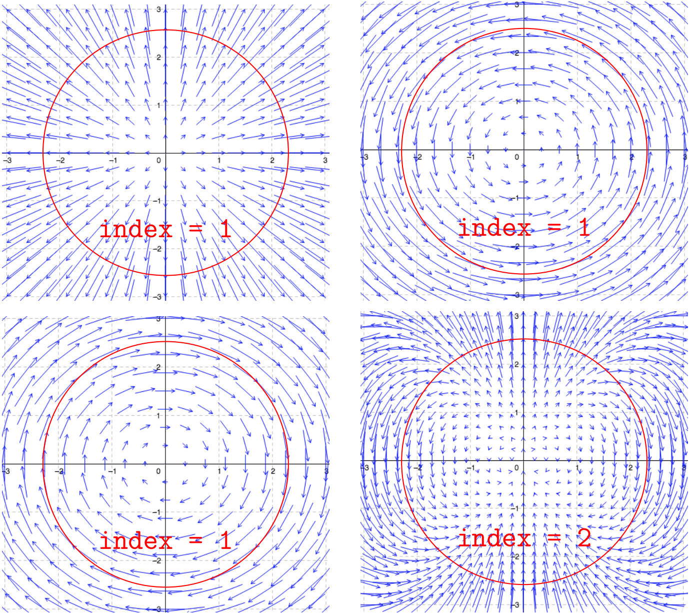
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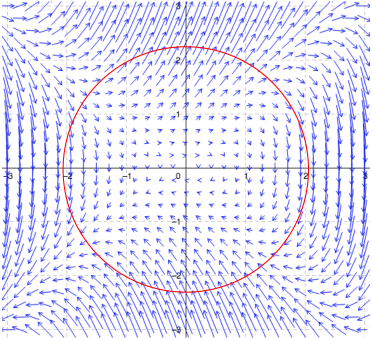

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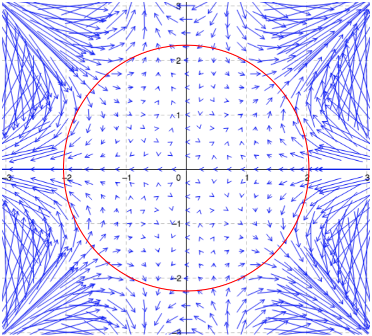

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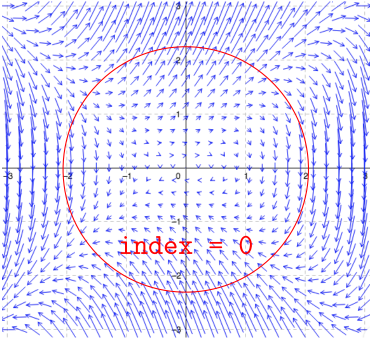
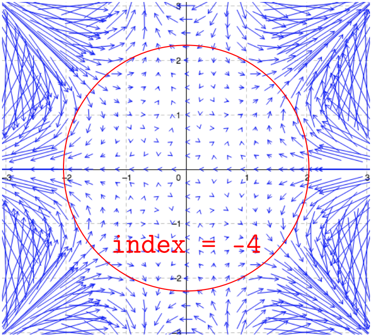
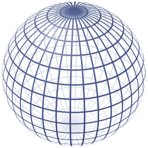

index = 0

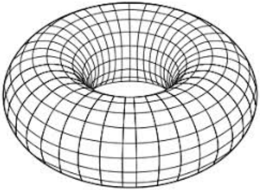

index = -4

The *Euler characteristic* of a shape M is the alternating sum

$$\chi(M) = \#\text{vertices} - \#\text{edges} + \#\text{faces}.$$

Examples:

$\chi(S^2) = \chi($  $) = 2.$

$\chi(\mathrm{torus}) = \chi($  $) = 0.$

The Poincaré-Hopf Theorem (1881, 1926)

If a vector field F on M has isolated zeroes, then

$$\chi(M) = \sum_{F(p)=0} \mathrm{index}(p).$$

In other words, the Euler characteristic is equal to the sum of the indices of all zero points for F.

Application: since $\chi(S^2) = 2$, any vector field F on S^2 must have some zeroes, in order for the sum on the right to be 2.

There cannot be a nonzero vector field on the two-dimensional sphere S^2!

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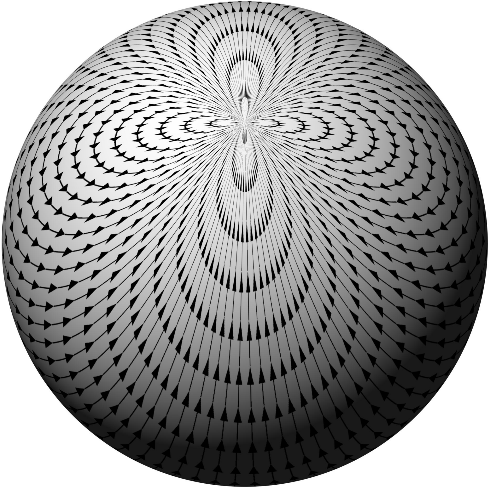
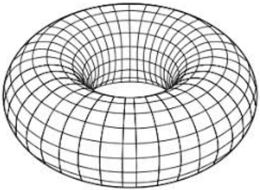


A vector field with a single zero of index 2.

Q: Can you find a nonzero vector field on the torus?



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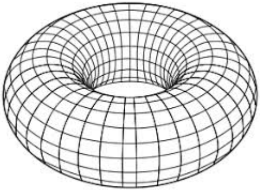
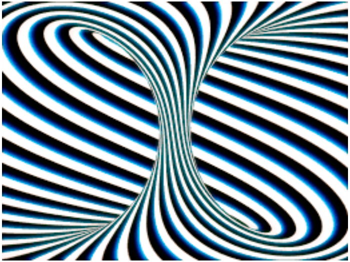
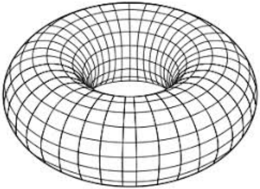
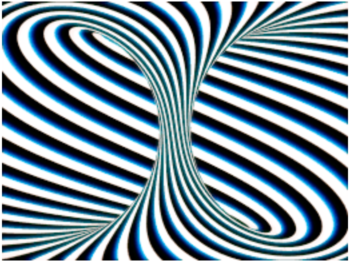




Yes, $\chi = 0$.
How about the three-holed torus?

Q: Can you find a nonzero vector field on the torus?





Yes, $\chi = 0$.

How about the three-holed torus?

No, $\chi = -4$.

Q: What about higher dimensional spheres?

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Slow your roll. Let's start with *lower* dimensional spheres. The one-dimensional sphere is the circle:

$$S^1 = \{(x, y) \in \mathbf{R}^2 \mid x^2 + y^2 = 1\}$$

There is a nonzero vector field on S^1 and $\chi(S^1) = 0$.

The *three-dimensional sphere* S^3 is the set of points in 4-space whose distance from the origin is one:

$$S^3 = \{(x_1, x_2, x_3, x_4) \in \mathbf{R}^4 \mid x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1\}$$

We can think of S^3 as three-space with an extra point ∞ "in all directions at once":

$$S^3 \cong \mathbf{R}^3 \cup \{\infty\}.$$

What is a vector field on S^3?

A vector field on S^3

Theorem (Hurwitz, Radon, Eckmann <1950's; Adams 1962)

Let $V(n)$ = number of linearly independent vector fields on the n-dimensional sphere S^n. Then,

$$V(n) = 0 \quad \textit{if } n \textit{ is even (we proved this for } S^2\textit{)}.$$

If n is odd:

n	1	3	5	7	9	11	13	15	17	19	21	23	...
$V(n)$	1	3	1	7	1	3	1	8	1	3	1	7	

For n odd, let 2^k be the largest power of 2 that divides $n+1$; writing $k = 4b + c$ where $0 \leq c \leq 3$, the number of linearly independent vector fields on S^n is:

$$V(n) = 8b + 2^c - 1.$$

The proof uses the dark arts of algebraic topology!