THE GROTHENDIECK TEICHMÜLLER GROUP (HIOB SEMINAR WS 2015-16)

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- *Talk 1: Introduction*

GT and the absolute Galois group

- *Talk 2: The étale fundamental group.* Recall the definition of an étale morphism. Define the étale fundamental group [10, I.§3,§5]. State Riemann's existence theorem [12, XII.5.1] and deduce the comparison theorem relating the étale fundamental group and the topological fundamental group [12, XII.5.2]. Deduce the isomorphism $\widehat{F_2} \cong \pi_1^{\text{ét}}(\mathbf{P}^1_{\overline{\mathbf{Q}}} \setminus \{0, 1, \infty\})$.

- *Talk 3: Curves and function fields.* Recall the equivalence between branched covers of normal proper curves and extensions of function fields. See [13, §4.4–§4.6], in particular Prop. 4.6.1, as well as [4, I.§6]. Establish the short exact sequence of profinite groups [13, Prop. 4.7.1]

$$(\star) \qquad 1 \longrightarrow \pi_1^{\text{ét}} X_{\overline{k}} \longrightarrow \pi_1^{\text{ét}} X \longrightarrow \text{Gal}(\overline{k}/k) \longrightarrow 1..$$

- *Talk 4: Belyi's Theorem I.* State Belyi's Theorem and prove the "if" part [13, see Remark 5.7.8].

- *Talk 5: Belyi's Theorem II.* Give the proof of the "only if" part of Belyi's theorem. Deduce that the monodromy action

$$\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \text{Out}(\pi_1^{\text{ét}}(\mathbf{P}^1_{\overline{\mathbf{Q}}} \setminus \{0, 1, \infty\}))$$

associated to the short exact sequence $(\star)$ is injective [13, 4.7.7 and 4.7.8].

- *Talk 6: Lifting the action of* $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$. The goal of this talk is to lift the monodromy action to give a faithful action of the absolute Galois group on $\widehat{F_2} \cong \pi_1^{\text{ét}}(\mathbf{P}^1_{\overline{\mathbf{Q}}} \setminus \{0, 1, \infty\})$. Give the construction of Ihara's homomorphism

$$\begin{aligned} \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) &\longrightarrow \widehat{\mathbf{Z}}^\times \times \widehat{F_2}' \\ \sigma &\longmapsto (\chi(\sigma), f_\sigma) \end{aligned}$$

from [11, §3.1]. Following [8, §1], deduce that the monodromy action lifts to give an injective homomorphism

$$(\dagger) \qquad \text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \text{Aut}(\widehat{F_2}).$$

- *Talk 7: The monomorphism* $\text{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) \longrightarrow \widehat{\text{GT}}$. The aim of this talk is to identify the image of the homomorphism $(\dagger)$ with the profinite Grothendieck-Teichmüller group $\widehat{\text{GT}}$. This consists in verifying relations (I), (II), and (III) for pairs $(\chi(\sigma), f_\sigma) \in \widehat{\mathbf{Z}}^\times \times \widehat{F_2}'$. Sketch the proof that these relations are satisfied, following [11, §3] and [8]. See also the appendix to [8] as well as [7].

The action of GT on the Teichmüller tower

- *Talk 8: Moduli spaces, the Teichmüller tower, and braid groups.* Introduce the Teichmüller space $\mathcal{T}_{g,n}$, the mapping class group $\Gamma_{g,n}$ and the moduli space $\mathcal{M}_{g,n} = \mathcal{T}_{g,n}/\Gamma_{g,n}$. Explain what it means that $\Gamma_{g,n}$ is the "orbifold fundamental group" of $\mathcal{M}_{g,n}$. Introduce the braid groups B_n, the pure braid groups K_n, the Hurwitz braid groups H_n, and prove that $\Gamma_{0,n}$ is isomorphic to the quotient $M(0, n)$ of H_n by its center [2]. In the remaining time, sketch the proof of Proposition A3 in the appendix of [9]. You should coordinate closely with (or be) the speaker for talk 9.

- *Talk 9:* $\widehat{\mathrm{GT}}$ *and braid groups.* Recall the action of $(\lambda, f) \in \widehat{\mathbf{Z}}^\times \times \widehat{F_2}'$ on $\widehat{F_2}$ from Talk 1. Show that the action extends to an automorphism of $\widehat{B_3}$ if and only if relations (I) and (II) are satisfied, then show that the action extends to an automorphism of the profinite completion $\widehat{M}(0,n)$ of the mapping class group if and only if relations (I) – (III) are satisfied. See [11, Lemmas 1 and 2] for an overview and [9, §3–§4] for full proofs. Reformulate these results in terms of the braid towers $\widehat{T}_n$ and prove the profinite case of the main theorem in [9] (see p. 12). The ideas here are in Drinfeld's original article [3] as well.

GT and the little discs operad

- *Talk 10: Operads, discs, braids.* Review the definition of an operad in a symmetric monoidal category. Define the operad of paranthesized braids $\mathcal{P}a\mathcal{B}$ in groupoids and explain its relationship with braided monoidal categories (see the paper [1] for a nice discussion of $\mathcal{P}a\mathcal{B}$). Define the little n-discs operad E_n and show that there is an equivalence of operads in spaces $B\mathcal{P}a\mathcal{B} \simeq E_2$ (there is a sketch of this fact in [14, §3.2]). References: [1] [5], [6], [14].

- *Talk 11: Profinite completion.* Introduce the category $\widehat{\mathbf{S}}$ of profinite spaces and the category $\widehat{\mathbf{G}}$ of profinite groupoids. Define Quick's model structure on $\widehat{\mathbf{S}}$ [15, 16] and Horel's model structure on $\widehat{\mathbf{G}}$ [6]. Show that in both instances the profinite completion functor is a left Quillen functor. Discuss the profinite classifying space functor $B\colon \widehat{\mathbf{G}} \longrightarrow \widehat{\mathbf{S}}$ and show that it is a Quillen right adjoint.

- *Talk 12:* $\widehat{\mathrm{GT}}$ *and the braid groups, revisited.* Prove that $\widehat{\mathrm{GT}}$ is the group of automorphisms of the profinite completion $\widehat{\mathcal{P}a\mathcal{B}}$ of the operad of paranthesized braids. This is a direct continuation of Talk 9. There is no reference in the literature for this proof–it amounts to explaining enough about the operad $\widehat{\mathcal{P}a\mathcal{B}}$ so that the result follows directly from the main theorem in [9].

- *Talk 13: Weak operads.* Review the notion of an algebraic theory. Give the definition of a weak operad and explain the model structure on the categories of operads and weak operads in both spaces and in groupoids. Explain the equivalence between weak operads and operads. Explain why $\mathcal{P}a\mathcal{U}\mathcal{B}$ is a cofibrant approximation of $\mathcal{P}a\mathcal{B}$. Reference: [6].

- *Talk 14:* $\widehat{\mathrm{GT}} \cong \mathrm{hAut}(\widehat{E_2})$. Prove the main theorem in [6].

References

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