Mathematics 311: Midterm Exam Solutions
John Lind

1. (5 points) Find all solutions $z \in \mathbf{C}$ to the equation $e^{2z} - 2e^z + 2 = 0$.

   Solution: Factor the left hand side as a quadratic expression in the entry e^z (if this seems like radically divine inspiration, notice that the quadratic formula leads to this choice of factorization)

   $$0 = e^{2z} - 2e^z + 2 = (e^z - (1+i))(e^z - (1-i)).$$

   Thus $e^z = 1 \pm i$, so $z = \log(1 \pm i)$. However, the complex logarithm is multiple-valued, so in order to find all solutions we need to express all of the values that $\log(1+i)$ and $\log(1-i)$ can take. Using the definition of the complex logarithm, this is:

   $$\begin{aligned} z = \log(1+i) &= \operatorname{Log}|1+i| + i\arg(1+i) \\ &= \operatorname{Log}\sqrt{2} + \frac{\pi i}{4} + k \cdot 2\pi i, \quad \text{where } k \text{ runs through all integers} \end{aligned}$$

   and

   $$\begin{aligned} z = \log(1-i) &= \operatorname{Log}|1-i| + i\arg(1-i) \\ &= \operatorname{Log}\sqrt{2} - \frac{\pi i}{4} + k \cdot 2\pi i, \quad \text{where } k \text{ runs through all integers} \end{aligned}$$

2. (5 points) Express $(i-1)^{2013}$ in the form $a + bi$ where $a, b \in \mathbf{R}$.

   Solution: It is very easy to answer this problem with

   $$(i-1)^{2013} = (\sqrt{2})^{2013}\cos(2013 \cdot 3\pi/4) + i(\sqrt{2})^{2013}\sin(2013 \cdot 3\pi/4).$$

   using the definition of complex exponentiation. I gave full credit for this response, because it technically answers the question, but I had a simpler form in mind. I should have asked you to find **integers** a and b!

   The trick is to write $i - 1$ in polar form before taking the exponent. Lo:

   $$(i-1)^{2013} = (\sqrt{2} \cdot e^{\frac{3\pi i}{4}})^{2013} = 2^{2013/2} \cdot e^{\frac{2013 \cdot 3\pi i}{4}}$$

   In the exponential, any multiple of 8 in the numerator will give an integer multiple of $2\pi i$, hence a factor of 1, so we only need to find the remainder of 2013 after dividing by 8. Long division shows that

   $$2013 = 8 \cdot 251 + 5,$$

so the exponent is:

$$\frac{2013 \cdot 3\pi i}{4} = \frac{(8 \cdot 251 + 5) \cdot 3\pi i}{4} = 2\pi i \cdot 251 \cdot 3 + \frac{15\pi i}{4}.$$

Returning to the original calculation and using the fact that $e^{2\pi i} = 1$, we have:

$$(i-1)^{2013} = 2^{2013/2} \cdot (e^{2\pi i})^{251 \cdot 3} \cdot e^{\frac{15\pi i}{4}} = 2^{1006} \cdot \sqrt{2} \cdot e^{\frac{15\pi i}{4}}.$$

The term $\sqrt{2} \cdot e^{\frac{15\pi i}{4}}$ is the polar form of $1 - i$, so this gives:

$$(i-1)^{2013} = 2^{1006}(1-i) = 2^{1006} - 2^{1006}i.$$

3. (5 points) Express the complex derivative f' of an analytic function f in terms of $\frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$, where $v = \text{Im} f$ is the imaginary part of f.

Solution: Write $f(z) = u(x,y) + iv(x,y)$ in terms of the real and imaginary components. The definition of the complex derivative at $z_0 = x_0 + iy_0$ is the limit

$$f'(z_0) = \lim_{w \to 0} \frac{f(z_0 + w) - f(z_0)}{w}.$$

Consider the limit for values of $w = x + iy$ approaching 0 along the real axis, i.e. for $y = 0$:

$$\begin{aligned} f'(z_0) &= \lim_{x \to 0} \frac{u(x_0 + x, y_0) + iv(x_0 + x, y_0) - u(x_0, y_0) - iv(x_0, y_0)}{x} \\ &= \lim_{x \to 0} \frac{u(x_0 + x, y_0) - u(x_0, y_0)}{x} + i \lim_{x \to 0} \frac{v(x_0 + x, y_0) - v(x_0, y_0)}{x} \\ &= \frac{\partial u}{\partial x}(x_0, y_0) + i \frac{\partial v}{\partial x}(x_0, y_0). \end{aligned}$$

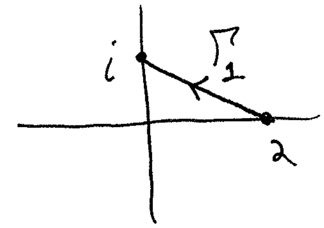
Notice that the final equality is simply the definition of the partial derivatives. The first Cauchy-Riemann equation is $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$. Making this substitution in the above expression, we find that

$$f' = \frac{\partial v}{\partial y} + i \frac{\partial v}{\partial x}$$

gives the complex derivative f' in terms of the partial derivatives of the imaginary component v. (You can also correctly derive this equation by taking the limit as w approaches 0 along the imaginary axis, yielding the equation $f' = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$, then applying the other Cauchy Riemann equation $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$.)

4. Compute the following contour integrals.

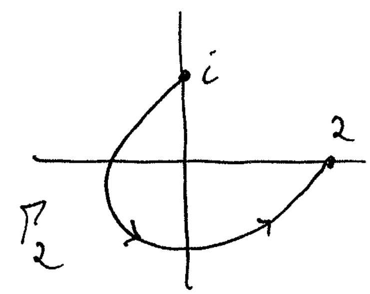
(a) (5 points) $\displaystyle\int_{\Gamma_1} \frac{1}{z}\,dz,$ where Γ_1 is the contour:



Solution: The contour Γ_1 avoids the negative real axis, so in the domain of integration the function $f(z) = 1/z$ has anti-derivative given by the principal branch of the logarithm $F(z) = \operatorname{Log}_{-\pi} z$. By the fundamental theorem of calculus for line integrals, this gives:

$$\begin{aligned}\int_{\Gamma_1} \frac{1}{z}\,dz = F(i) - F(2) &= \operatorname{Log}_{-\pi}(i) - \operatorname{Log}_{-\pi}(2) \\ &= \operatorname{Log}|i| + i\operatorname{Arg}_{-\pi}(i) - \operatorname{Log}|2| - i\operatorname{Arg}_{-\pi}(2) \\ &= \operatorname{Log} 1 + i \cdot \frac{\pi}{2} - \operatorname{Log} 2 - i \cdot 0 \\ &= \frac{\pi i}{2} - \operatorname{Log} 2\end{aligned}$$

(b) (5 points) $\displaystyle\int_{\Gamma_2} \frac{1}{z}\,dz,$ where Γ_2 is the contour:



Solution: The contour Γ_2 avoids the ray emanating from the origin at angle $\theta = \pi/4$, so in the domain of integration the function $f(z) = 1/z$ has anti-derivative given by the branch of the logarithm $F(z) = \operatorname{Log}_{\pi/4} z$. By the fundamental theorem of calculus for line integrals, this gives:

$$\begin{aligned}\int_{\Gamma_2} \frac{1}{z}\,dz = F(2) - F(i) &= \operatorname{Log}_{\pi/4}(2) - \operatorname{Log}_{\pi/4}(i) \\ &= \operatorname{Log}|2| + i\operatorname{Arg}_{\pi/4}(2) - \operatorname{Log}|i| - i\operatorname{Arg}_{\pi/4}(i) \\ &= \operatorname{Log} 2 + i \cdot 2\pi - \operatorname{Log} 1 - i \cdot \frac{\pi}{2} \\ &= \operatorname{Log} 2 + \frac{3\pi i}{2}\end{aligned}$$

Notice that the closed loop $\Gamma_1 \boxplus \Gamma_2$ circles the origin once with positive orientation. Thus by Cauchy's theorem and the basic calculation for line integrals,

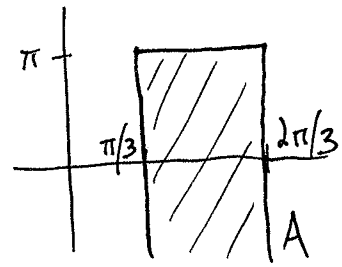
$$\int_{\Gamma_1} \frac{1}{z}\,dz + \int_{\Gamma_2} \frac{1}{z}\,dz = \int_{\Gamma_1 \boxplus \Gamma_2} \frac{1}{z}\,dz = 2\pi i. \tag{1}$$

This calculation agrees with the two calculations above. In fact, an alternative solution is to compute either (a) or (b), then deduce the other calculation using equation (1).

5. Consider the complex function $f(z) = e^{iz}$.

(a) (5 points) Let A be the set of complex numbers $z = x + iy$ satisfying the inequalities $\pi/3 \le x \le 2\pi/3$ and $y \le \pi$. Describe the image $f(A)$ of the set A under the function f. It may be useful to draw pictures, but please give an explicit written answer to accompany any images.

Solution: Here is a picture of the set A.



Notice that the function $f(z) = e^{iz}$ is NOT the exponential function e^z. The function

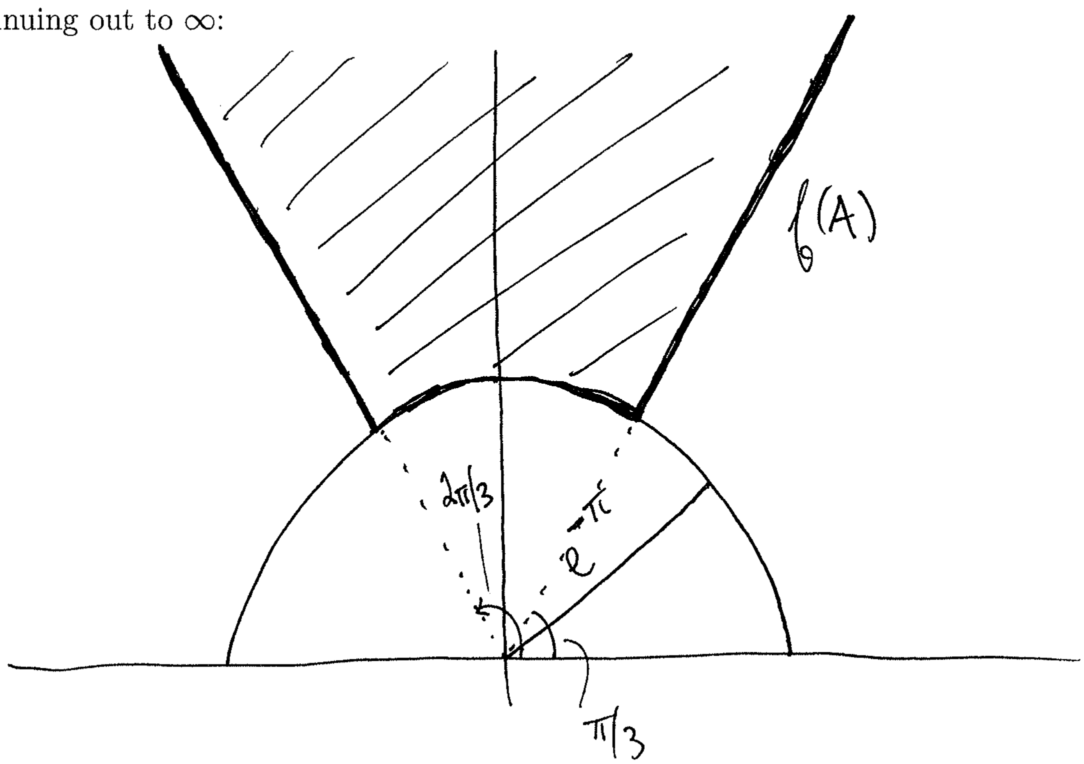
$$f(z) = f(x + iy) = e^{-y} \cdot e^{ix}$$

wraps the real axis onto the unit circle in the counter clockwise direction with period 2π and takes the horizontal line $y = y_0$ onto the circle centered at the origin of radius e^{-y_0}. Another way of thinking about f is that it takes the vertical line $x = x_0$ to the ray emanating from the origin at the angle x_0.

As $z = x + iy$ runs through the points in A, the output $f(z) = e^{-y} \cdot e^{ix} = r \cdot e^{i\theta}$ runs through points of distance $r \ge e^{-\pi}$ from the origin and at an angle θ between $\pi/3$ and $2\pi/3$. Thus f takes A to the set (expressed in polar coordinates):

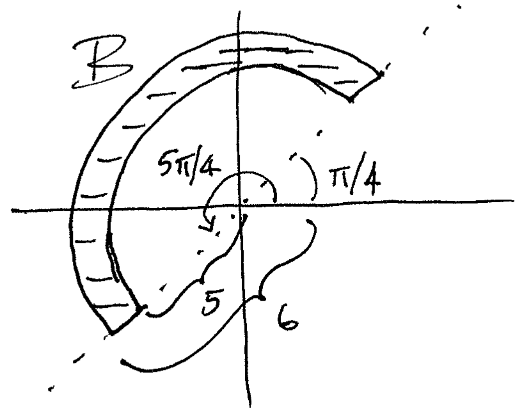
$$f(A) = \{re^{i\theta} \mid r \ge e^{-\pi} \text{ and } \pi/3 \le \theta \le 2\pi/3\}.$$

This is the sector bounded by the angles $\pi/3$ and $2\pi/3$, starting at radius $e^{-\pi}$ and continuing out to ∞:



(b) (5 points) Consider the semi-annulus B described in polar coordinates as those points (r, θ) that satisfy $5 \leq r \leq 6$ and $\pi/4 \leq \theta \leq 5\pi/4$. Find a subset of the complex plane that is mapped by the function f onto B.

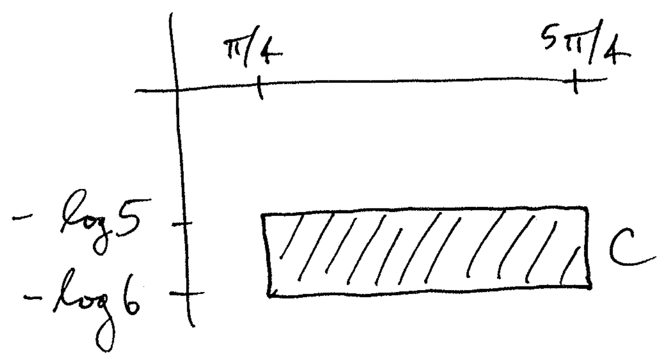
Solution: Here is a picture of the set B.



In order for $f(z) = e^{-y} \cdot e^{ix} = r \cdot e^{i\theta}$ to hit all of the points in B, we must have x and y values that achieve all values within the inequalities $5 \leq e^{-y} \leq 6$ and $\pi/4 \leq x \leq 5\pi/4$. Thus the set

$$C = \{x + iy \mid \pi/4 \leq x \leq 5\pi/4 \text{ and } -\operatorname{Log} 6 \leq y \leq -\operatorname{Log} 5\}$$

is sent onto the set B under the function f. The set C is the rectangle pictured below:



(Other possible solutions are the various translates of C by integer multiples of 2π in the real direction.)

6. (15 points) Consider the function $f(z) = \dfrac{4z^2 + 6iz - 2}{z^3 - z^2 + z - 1}$.

(a) Determine the domain D of analyticity of f (i.e. the largest domain in the complex plane on which $f(z)$ is an analytic function).

Solution: We may factor f and cancel the $(z+i)$ terms from the numerator and denominator:

$$f(z) = \frac{4z^2 + 6iz - 2}{z^3 - z^2 + z - 1} = \frac{(z+i)(4z+2i)}{(z+i)(z-i)(z-1)} = \frac{4z+2i}{(z-i)(z-1)}.$$

The resulting rational function is in reduced form (there are no further cancellations), so its poles occur ar $z = i, 1$. Thus the domain of analyticity is $D = \mathbf{C} - \{i, 1\}$, the set of all complex numbers $z \neq i, 1$.

(b) Write down the partial fraction decomposition of $f(z)$.

Solution: We seek to write $f(z)$ in the form

$$f(z) = \frac{4z+2i}{(z-i)(z-1)} = \frac{A}{z-i} + \frac{B}{z-1}.$$

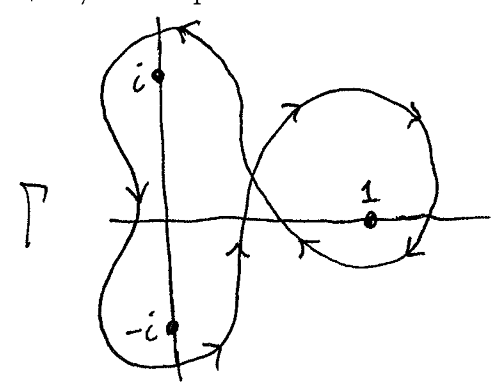
Combining the fractions on the right, we arrive at the equation

$$A(z-1) + B(z-i) = 4z + 2i.$$

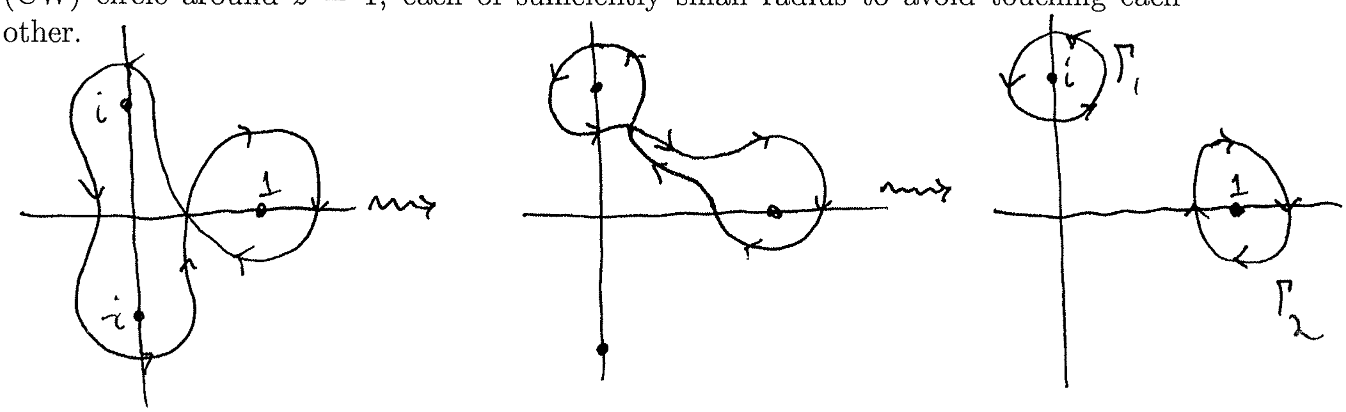
Solving for A and B, we find that $A = 3 - 3i$ and $B = 1 + 3i$, so the partial fraction decomposition is:

$$f(z) = \frac{3-3i}{z-i} + \frac{1+3i}{z-1}.$$

(c) Compute $\displaystyle\int_\Gamma f(z)\,dz$, where Γ is the contour:



Solution: Noting that $f(z)$ is analytic at $z = -i$, we see that there is a homotopy within the domain of analyticity of f from the contour Γ to the contour $\Gamma_1 \boxplus \Gamma_2$, where Γ_1 is a positively oriented (CCW) circle around $z = i$ and Γ_2 is a negatively oriented (CW) circle around $z = 1$, each of sufficiently small radius to avoid touching each other.



We may now compute the integral using Cauchy's integral theorem and the basic calculation for integrals around positively oriented circles γ.

$$\int_\gamma \frac{1}{z-a}\,dz = \begin{cases} 0 & \text{if } a \text{ lies outside of } \gamma \\ 2\pi i & \text{if } a \text{ lies inside of } \gamma \end{cases}$$

The calculation is:

$$\begin{aligned}
\int_\Gamma f(z)\,dz &= \int_{\Gamma_1} \left(\frac{3-3i}{z-i} + \frac{1+3i}{z-1}\right) dz + \int_{\Gamma_2} \left(\frac{3-3i}{z-i} + \frac{1+3i}{z-1}\right) dz \\
&= (3-3i)\int_{\Gamma_1} \frac{1}{z-i}\,dz + (1+3i)\int_{\Gamma_1} \frac{1}{z-1}\,dz \\
&\qquad + (3-3i)\int_{\Gamma_2} \frac{1}{z-i}\,dz + (1+3i)\int_{\Gamma_2} \frac{1}{z-1}\,dz \\
&= (3-3i)\cdot 2\pi i + (1+3i)\cdot 0 + (3-3i)\cdot 0 + (1+3i)\cdot(-2\pi i) \\
&= 12\pi + 4\pi i
\end{aligned}$$

Notice that the second integral around Γ_2 contributes a minus sign because Γ_2 is negatively oriented.