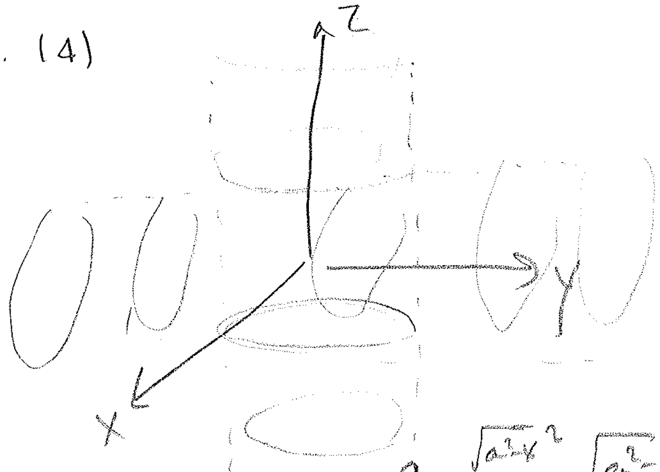
6. (4)



$$-a \le x \le a$$

$$-\sqrt{a^2-y^2} \le y \le \sqrt{a^2-x^2}$$

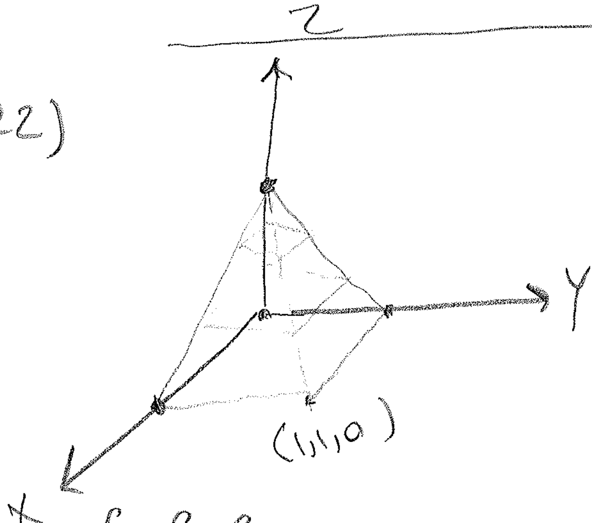
$$-\sqrt{a^2-x^2} \le z \le \sqrt{a^2-x^2}$$

$$V = \int_{-a}^{a} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} \int_{-\sqrt{a^2-x^2}}^{\sqrt{a^2-x^2}} dz\, dy\, dx$$

$$= \int_{-a}^{a} 4(a^2-x^2)\, dx = 4a^2x \Big|_{-a}^{a} - \frac{4}{3}x^3 \Big|_{-a}^{a}$$

$$= \boxed{8a^3 - \frac{8}{3}a^3} = \boxed{\frac{16}{3}a^3}$$

22)



our region can be interpreted by

$0 \le z \le 1$, $0 \le x+z \le 1$, $0 \le y+z \le 1$, $x \ge 0$, $y \ge 0$

so

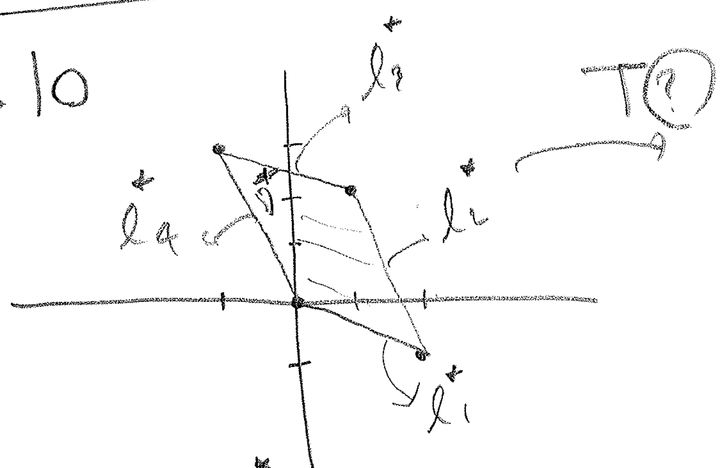
$$\iiint_W (x^2+y^2)\, dx\, dy\, dz = \int_0^1 \int_0^{1-z} \int_0^{1-z} (x^2+y^2)\, dx\, dy\, dz =$$

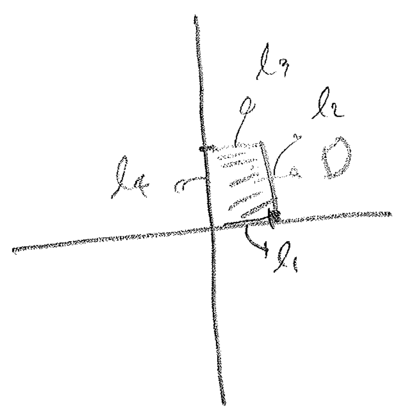
$$22) \Rightarrow \int_0^1 \left(\int_0^{1-z} \left(\frac{x^3}{3} + xy^2 \right) \Big|_0^{1-z} dy \right) dz$$

$$= \int_0^1 \int_0^{1-z} \left(\frac{(1-z)^3}{3} + (1-z)y^2 \right) dy = \int_0^1 2\,\frac{(1-z)^4}{3} dz$$

$$\overset{u=1-z}{=} +\frac{2}{3}\int_0^1 u^4\, du = \boxed{\frac{2}{15}}$$

6.1.10





Equation for l_1^* is:

$$y^* = -\frac{1}{2}x^*$$

and equatin for l_4^* is:

$$y^* = -3x^*$$

$$\begin{cases} T(l_1^*) = l_1 = \{(x,y) : y=0,\ 0 \le x \le 1\} \\ T(l_4^*) = l_4 = \{(x,y) : 0 \le y \le 1,\ x=0\} \end{cases}$$

$\Rightarrow$ 6.1.10)

To be more precise, if we let $T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then

$$\begin{cases} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} 2a - b = 1 \\ 2c - d = 0 \end{cases} \\ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \begin{cases} -a + 3b = 0 \\ 3d - c = 1 \end{cases} \end{cases}$$

so $a = \frac{3}{5}$, $b = 1/5$, $c = \frac{1}{5}$, $d = \frac{2}{5}$

$$\rightarrow T = \frac{1}{5} \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

6.1.14) $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$T(q) = T(p) + T(\lambda v) + T(\mu w)$$

$$= T(p) + \lambda T(v) + \mu T(w)$$

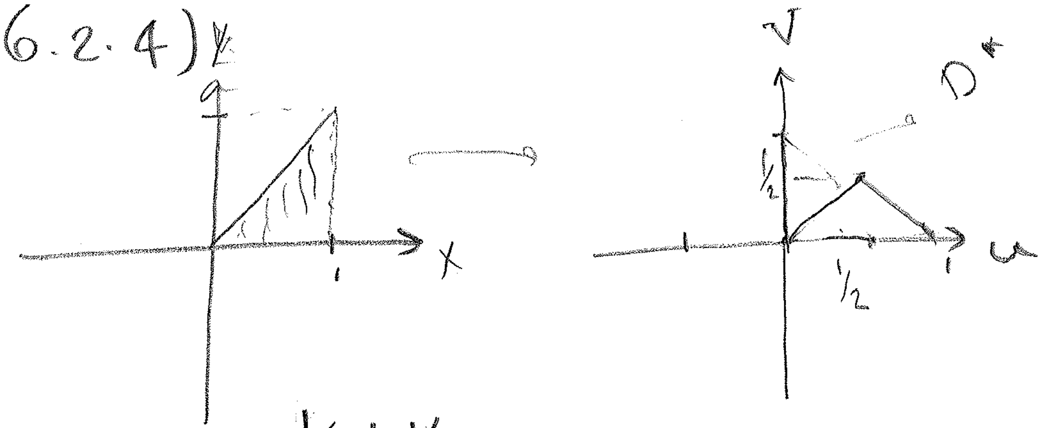
$$= \tilde{p} + \lambda \tilde{v} + \mu \tilde{w}$$

$\det(T) \neq 0 \Rightarrow \tilde{v} \neq 0, \tilde{w} \neq 0$

and also $\tilde{v}$ is not a multip scalar of $\tilde{w}$, because if:

$$\tilde{v} = r\tilde{w} \Rightarrow \begin{cases} T^{-1}(r\tilde{w}) = r T^{-1}(\tilde{w}) = rw \\ T^{-1}(\tilde{v}) = v \end{cases} \Rightarrow v = rw \text{ (Contradiction)}$$

6.2.4)



$$D^* = \begin{cases} 0 \le v \le \frac{1}{2} \\ v \le u \le 1-v \end{cases}$$

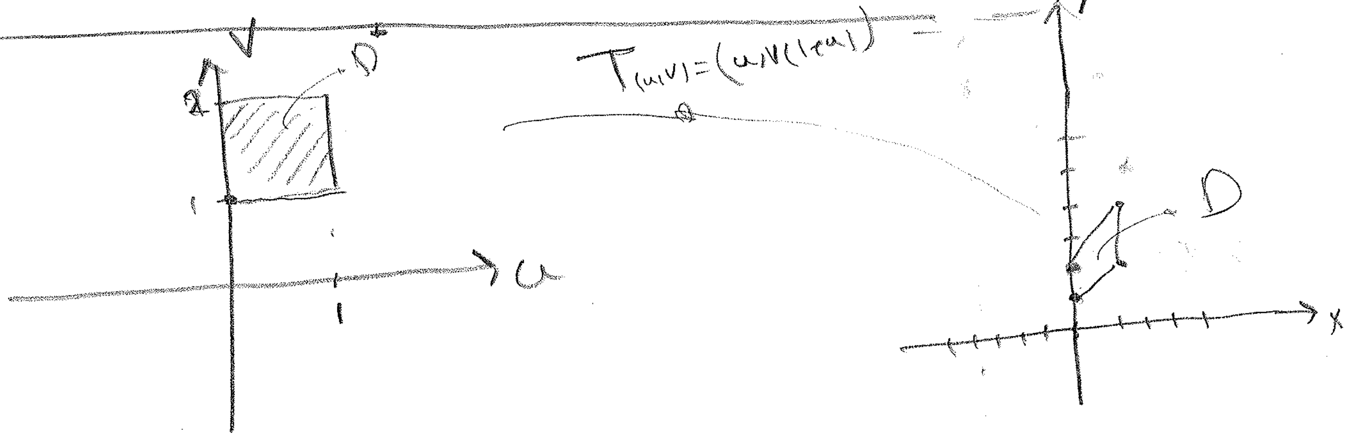
$$\left|\frac{\partial(x,y)}{\partial(v,u)}\right| = \left|\begin{matrix} 1 & 1 \\ -1 & +1 \end{matrix}\right| = +2$$

$$\iint_D (x+y)\,dx\,dy = \int_0^{1/2}\int_v^{1-v} (2u)\left|\frac{\partial(x,y)}{\partial(v,u)}\right| du\,dv = +4\int_0^{1/2}\int_v^{(1-v)} u\,dv\,du$$

$$= +4\int_0^{\frac{1}{2}} \frac{u^2}{2}\Big|_v^{1-v} dv = 2\int_0^{1/2}\left((1-v)^2 - v^2\right) dv$$

$$= 2\int_0^{1/2} (1-2v)\,dv = 2\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{1}{2}$$

6.2.6



$T(0,1) = (0,1)$, $T(1,1) = (1,2)$, $T(1,2) = (1, 2\times 2) = (1,4)$

$T(0,2) = (0,2)$

6.2.6 a)

$$\iint_D xy\,dx\,dy = \iint_{D^*} u\,(v(1+u))\left|\frac{\partial(x,y)}{\partial(u,v)}\right| dv\,du$$

$$= +\int_0^1\int_1^2 (uv+u^2v)\left|\begin{matrix}1 & 0\\ v & 1+u\end{matrix}\right| dv\,du$$

$$= +\int_1^2\int_0^1 (uv+u^2v)(1+u)\,du\,dv = \int_1^2 \left(\frac{u^4}{4}+\frac{2}{3}u^3+\frac{u^2}{2}\right)v\Big|_0^1 dv$$

$$= +\int_1^2\left(\frac{1}{4}+\frac{2}{3}+\frac{1}{2}\right)v\,dv = \frac{3}{2}\cdot\frac{17}{12} = \boxed{\frac{17}{8}}$$

6.2.6 b)

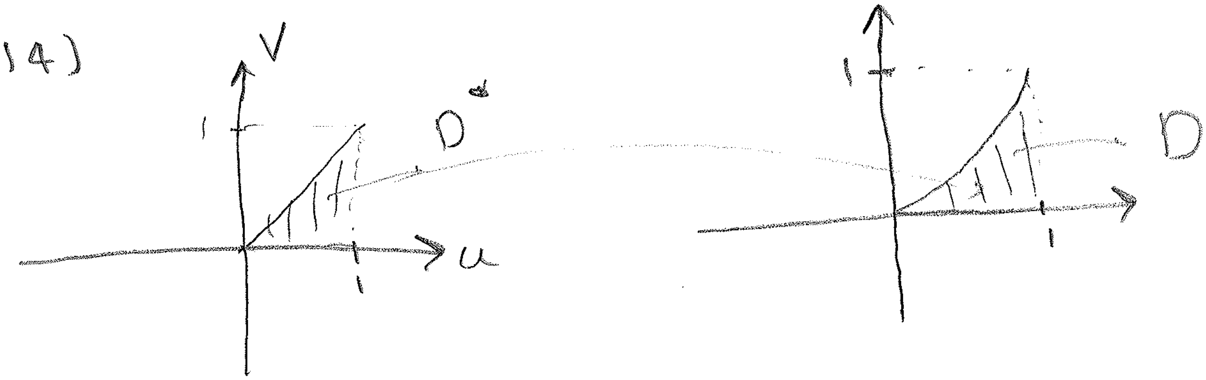
$$\iint_D (x-y)\,dx\,dy = \iint_{D^*}(u-v-uv)\overbrace{\left|\frac{\partial(x,y)}{\partial(v,u)}\right|}^{(1+u)} dv\,du$$

$$= \int_1^2\int_0^1 (u-v-uv)(u+1)\,du\,dv = \int_1^2\int_0^1 (u^2+u-vu-vu^2-uv)\,du\,dv$$

$$= \int_1^2\left(\frac{u^3}{3}+\frac{u^2}{2}-v\frac{u^2}{2}-v\frac{u^3}{3}-\frac{u^2}{2}v\right)\Big|_0^1 dv = \int_1^2\left(\frac{1}{3}+\frac{1}{2}-v-\frac{v}{3}\right)dv = \frac{5}{6}v-\frac{4}{3}\frac{v^2}{2}\Big|_1^2$$

$$= \left(\frac{5}{6}\times 2-\frac{4}{3}\cdot\frac{4}{2}\right)-\left(\frac{5}{6}-\frac{4}{3}\cdot\frac{1}{2}\right) = \frac{5}{6}-\frac{12}{6} = \boxed{\frac{-7}{6}}$$

6.2.14)



Define $T: \mathbb{R}^2 \to \mathbb{R}^2$

$T(u,v) = (u^2, v)$,

i) $T(D^*) = D$

ii) T is one-to-one

proof: if $T(u_1,v_1) = T(u_2,v_2) \Leftrightarrow (u_1^2, v_1) = (u_2^2, v_2) \Leftrightarrow \begin{cases} u_1^2 = u_2^2 \\ v_1 = v_2 \end{cases}$

$\Leftrightarrow \begin{cases} u_1 = u_2 \text{ or } u_1 = -u_2 \\ v_1 = v_2 \end{cases} \Leftrightarrow \begin{cases} u_1 = u_2 \text{ — because } u \geq 0 \\ v_1 = v_2 \end{cases}$

$\left| \frac{\partial(x,y)}{\partial(u,v)} \right| = \begin{pmatrix} 2u & 0 \\ 0 & v \end{pmatrix} = \boxed{2uv}$

a), b) $\int_0^1 \int_0^{x^2} xy \, dy \, dx = \int_0^1 \int_0^u u^2 v \, dv \, du = \int_0^1 \frac{u^4}{2} du = \boxed{\frac{1}{10}}$

6.2.17

$x = r\cos(\theta)$, $y = r\sin(\theta)$

$$r^4 = 2a^2 r^2(\cos^2 - \sin^2) = 2a^2 r^2(\cos(2\theta))$$

$$\Rightarrow r^2 = 2a^2(\cos(2\theta))$$

$$\text{Area} = \int_0^{\pi/4}\int_0^{\sqrt{2a^2\cos(2\theta)}} r\,dr\,d\theta = \int_0^{\pi/4} \frac{r^2}{2}\Big|_0^{\sqrt{2a^2\cos(2\theta)}} d\theta$$

$$= \frac{2a^2}{2}\int_0^{\pi/4} \cos(2\theta)\,d\theta = a^2 \frac{\sin(2\theta)}{2}\Big|_0^{\pi/4}$$

$$= \boxed{\frac{a^2}{2}}$$

Because it is symmetric respect to x-axis:

So

$$\text{Area} = 2\cdot\frac{a^2}{2} = \boxed{a^2}$$

6.2.26)

In the polar coordinates we have

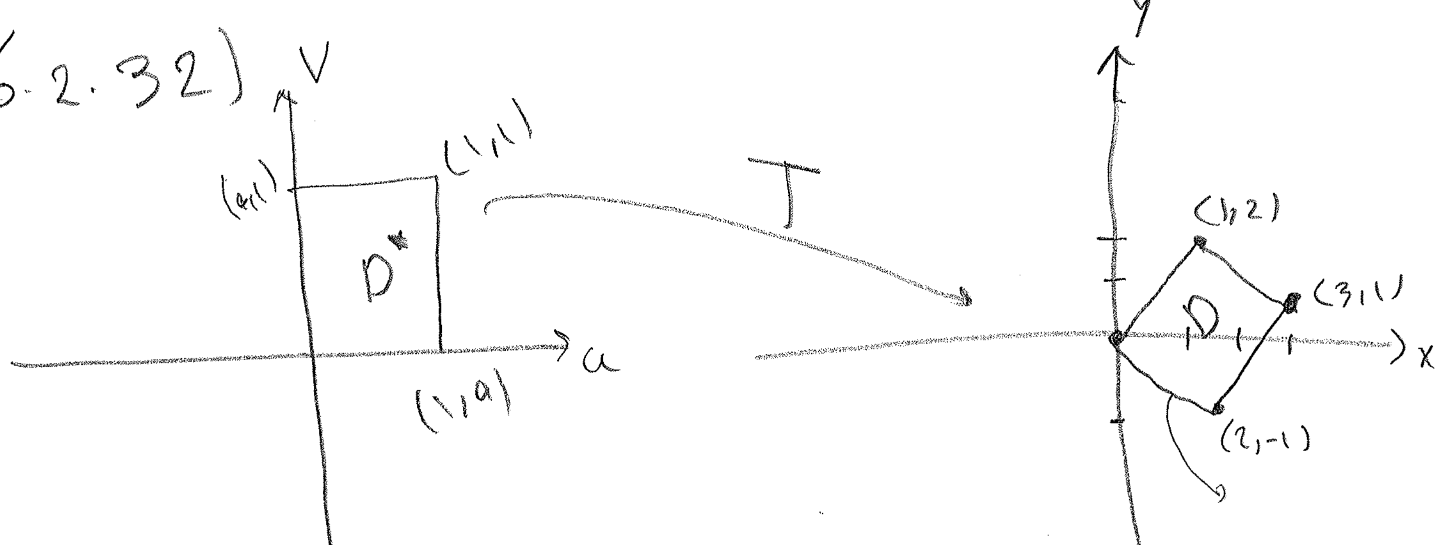
$0 \le \rho \le 3,\ 0 \le \theta \le \pi/2\ \ 0 \le \phi \le \pi/2$

$$A = \int_0^3 \int_0^{\pi/2} \int_0^{\pi/2} \frac{\rho}{1+\rho^4}\, \rho^2 \sin(\phi)\, d\phi\, d\theta\, d\rho$$

$$= \frac{\pi}{2} \int_0^3 \frac{\rho^3}{1+\rho^4}\, d\rho = \frac{\pi}{2} \cdot \frac{1}{4} \ln(1+\rho^4) \Big|_0^3$$

$$= \boxed{\frac{\pi}{8} \ln(82)}$$

6.2.32)



6.2.32)

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \to \begin{cases} a = 2 \\ c = -1 \end{cases}$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \to \begin{bmatrix} b \\ d \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \to \begin{cases} b = 1 \\ d = 2 \end{cases}$$

$$\Rightarrow T = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}$$

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \left|\begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}\right| = 5$$

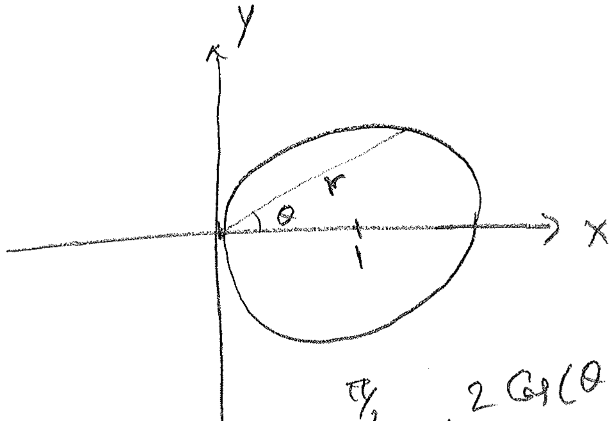
$$x + y = (2u + v) + (-u + 2v) = u + 3v$$

$$\iint_D (x+y)\,dx\,dy = \int_0^1\int_0^1 (u+3v)\,5\,du\,dv$$

$$= \boxed{10}$$

6.3.10

$$\begin{cases} x^2+y^2=2x \\ x^2+y^2=z^2 \end{cases} \simeq \begin{cases} (x-1)^2+y^2=1 \\ x^2+y^2=z^2 \end{cases}, \quad \delta=\sqrt{x^2+y^2}$$



$$\rightarrow \begin{cases} x=r\cos\theta \\ y=r\sin\theta \end{cases}, \quad -\pi/2 \le \theta \le \pi/2, \quad 0 \le r \le 2\cos\theta$$

$$z^2=r^2 \Rightarrow |z|=r$$

$$\Rightarrow \text{Mass} = \int_{-\pi/2}^{\pi/2}\int_0^{2\cos(\theta)}\int_{-r}^{r} r\, r\,dz\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\left(\int_0^{2\cos(\theta)} 2r^3\,dr\right)d\theta = \int_{-\pi/2}^{\pi/2} \frac{r^4}{2}\Big|_0^{2\cos\theta}$$

$$= \int_{-\pi/2}^{\pi/2} 8\cos^4\theta\,d\theta = 8\int_{-\pi/2}^{\pi/2}\left(\frac{1+\cos(2\theta)}{2}\right)^2 d\theta = 2\left(\int_{-\pi/2}^{\pi/2} d\theta + \int 2\cos 2\theta\,d\theta + \int_{-\pi/2}^{\pi/2}\cos^2 2\theta \ldots\right.$$

$$= 2\pi + 2\int_{-\pi/2}^{\pi/2}\frac{1+\cos(4\theta)}{2}\,d\theta = \boxed{3\pi}$$

6.3.16

Let $\begin{cases} x = \rho \sin\phi \cos(\theta) \\ y = \rho \sin(\phi) \sin(\theta) \\ z = \rho \cos(\phi) \end{cases}$, $0 \le \theta \le 2\pi$, $0 \le \rho \le 1$, $0 \le \phi \le \pi$

$$\iiint_W e^{-z}\,dx\,dy\,dz = \int_0^1 \int_0^{2\pi} \int_0^{\pi} e^{-\rho\cos(\phi)} \rho^2 \sin(\phi)\,d\phi\,d\theta\,d\rho$$

$$= \int_0^1 \int_0^{2\pi} \rho e^{-\rho\cos(\phi)} \Big|_0^{\pi} d\theta\,d\rho = \int_0^1 \int_0^{2\pi} (\rho e^{\rho} - \rho e^{-\rho})\,d\theta\,d\rho$$

$$= 2\pi \int_0^1 (\rho e^{\rho} - \rho e^{-\rho})\,d\rho = 2\pi\left(\rho e^{\rho}\Big|_0^1 - \int_0^1 e^{\rho}\,d\rho \right) - 2\pi\left(-\rho e^{-\rho}\Big|_0^1 + \int_0^1 e^{-\rho}\,d\rho\right)$$

$$= 2\pi\left(e^1 - (e^1 - 1)\right) - 2\pi\left(-e^{-1} + e^{-1} - 1\right)$$

$$= 2\pi + 2\pi = 4\pi$$

average value of $e^{-z} = \dfrac{4\pi}{\frac{4}{3}\pi} = \boxed{3}$