HW #8

§5.2

2.) c.) $R=[0,1]\times[0,1]$

$$\iint_R \sin(x+y)\,dx\,dy = \int_0^1\int_0^1 \sin(x+y)\,dx\,dy$$

$$= \int_0^1 -\cos(x+y)\Big|_{y=0}^{y=1} dy$$

$$= \int_0^1 -\cos(x+1) + \cos(x)\,dy$$

$$= -\sin(x+1)\Big|_{x=0}^{x=1} + \sin(x)\Big|_{x=0}^{x=1}$$

$$= -\sin(2)+\sin(1)+\sin(1)-\sin(0)$$

$$= \boxed{2\sin(1)-\sin(2)} ✓$$

2.) d.) $\iint_R (x^2+2xy+y\sqrt{x})\,dx\,dy$

$$= \int_0^1\int_0^1 x^2+2xy+y\sqrt{x}\,dx\,dy = \int_0^1 \tfrac{1}{3}x^3+x^2y+\tfrac{2}{3}yx^{3/2}\Big|_{x=0}^{x=1} dy$$

$$= \int_0^1 \left(\tfrac{1}{3}+y+\tfrac{2}{3}y\right)-(0)\,dy = \int_0^1 \tfrac{1}{3}+y+\tfrac{2}{3}y\,dy$$

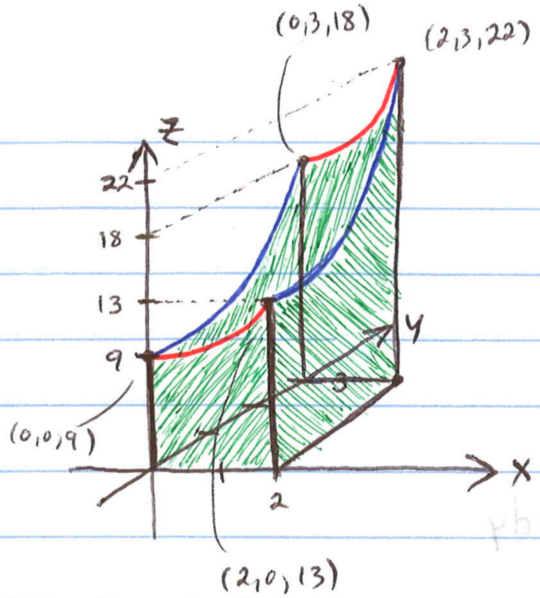
$$= \tfrac{1}{3}y+\tfrac{1}{2}y^2+\tfrac{1}{3}y^2\Big|_{y=0}^{y=1} = \tfrac{1}{3}+\tfrac{1}{2}+\tfrac{1}{3} = \boxed{\tfrac{7}{6}} ✓$$

6.) $\int_0^3\int_0^2 (9+x^2+y^2)\,dx\,dy$, $\begin{cases} 0\le x\le 2 \\ 0\le y\le 3\end{cases}$

$z=f(x,y)=9+x^2+y^2$

$f(0,0)=9$, $f(0,3)=18$

$f(2,0)=13$, $f(2,3)=22$



$f(0,y) = 9 + y^2$

$f(x,0) = 9 + x^2$

$f(2,y) = 13 + y^2$

$f(x,3) = 18 + y^2$

8.) $\begin{cases} xz\text{-plane} & (y=0) \\ yz\text{-plane} & (x=0) \\ xy\text{-plane} & (z=0) \\ x=1 & \\ y=1 & \\ z = x^2+y^4 = f(x,y) & \end{cases}$

$$\int_0^1\int_0^1 f(x,y)\,dx\,dy = \int_0^1\int_0^1 x^2+y^4\,dx\,dy = \int_0^1 \frac{1}{3}x^3 + xy^4\Big|_{x=0}^{x=1} dy = \int_0^1 \frac{1}{3}+y^4\,dy$$

$$= \frac{1}{3}y + \frac{1}{5}y^5\Big|_{y=0}^{y=1} = \frac{1}{3}+\frac{1}{5} = \boxed{\frac{8}{15}} \checkmark$$

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§5.3 4.) a.) $\displaystyle\int_{-3}^{2}\int_0^{y^2}(x^2+y)\,dx\,dy = \int_{-3}^{2}\frac{1}{3}x^3+xy\Big|_{x=0}^{x=y^2}dy$

$$= \int_{-3}^{2}\frac{1}{3}(y^2)^3+(y^2)y\,dy$$

$$= \int_{-3}^{2}\frac{1}{3}y^6+y^3\,dy = \frac{1}{21}y^7+\frac{1}{4}y^4\Big|_{y=-3}^{y=2}$$

$$= \frac{1}{21}(2)^7+\frac{1}{4}(2)^4-\frac{1}{21}(-3)^7-\frac{1}{4}(-3)^4 = \boxed{\frac{7895}{84}} \checkmark$$

4.) d.) $\displaystyle\int_0^{\pi/2}\int_0^{\cos x} y\sin x\, dy\, dx = \int_0^{\pi/2} \tfrac{1}{2}y^2 \sin x \Big|_{y=0}^{y=\cos x} dx$

$$= \int_0^{\pi/2} \tfrac{1}{2}\cos^2 x \sin x\, dx$$

Let $u = \cos x$

$\frac{du}{dx} = -\sin x$

$\Rightarrow -\frac{du}{dx} = \sin x$

$\Rightarrow -du = \boxed{\sin x\, dx} \Rightarrow = -\frac{1}{2}\displaystyle\int_{\cos(0)=1}^{\cos(\pi/2)=0} u^2\, du$

$$= -\tfrac{1}{2}\left(\tfrac{1}{3}u^3\Big|_1^0\right)$$

$$= \tfrac{1}{2}\left(\tfrac{1}{3}\right)$$

$$= \boxed{\tfrac{1}{6}} \checkmark$$

4.) f.) $\displaystyle\int_{-1}^{0}\int_0^{2(1-x^2)^{1/2}} x\, dy\, dx =$

$$= \int_{-1}^{0} xy \Big|_{y=0}^{y=2\sqrt{1-x^2}} dx = \int_{-1}^{0} 2x\sqrt{1-x^2}\, dx$$

Let $u = 1-x^2$

$\frac{du}{dx} = -2x$

$\Rightarrow -\frac{du}{dx} = 2x$

$\Rightarrow -du = \boxed{2x\, dx} \Rightarrow = -\displaystyle\int_{1-(-1)^2}^{1-(0)^2} \sqrt{u}\, du = -\int_0^1 \sqrt{u}\, du$

$$= -\tfrac{2}{3}u^{3/2}\Big|_{u=0}^{u=1} = \boxed{-\tfrac{2}{3}} \checkmark$$

12.) $\displaystyle\iint_D \cos y\, dx\, dy$, *bounded by* $y = 2x$, $y = x$, $x = \pi$, $x = 2\pi$

$$= \int_{\pi}^{2\pi}\int_{x}^{2x} \cos(y)\,dy\,dx = \int_{\pi}^{2\pi} \sin(y)\Big|_{y=x}^{y=2x} dx$$

$$= \int_{\pi}^{2\pi} \sin(2x) - \sin(x)\,dx$$

$$= -\tfrac{1}{2}\cos(2x)\Big|_{x=\pi}^{x=2\pi} + \cos(x)\Big|_{x=\pi}^{x=2\pi}$$

$$= -\tfrac{1}{2}\cos(4\pi) + \tfrac{1}{2}\cos(2\pi) + \cos(2\pi) - \cos(\pi)$$

$$= \cancel{-\tfrac{1}{2}} \cancel{+\tfrac{1}{2}} + 1 - (-1)$$

$$= \boxed{2} \checkmark$$

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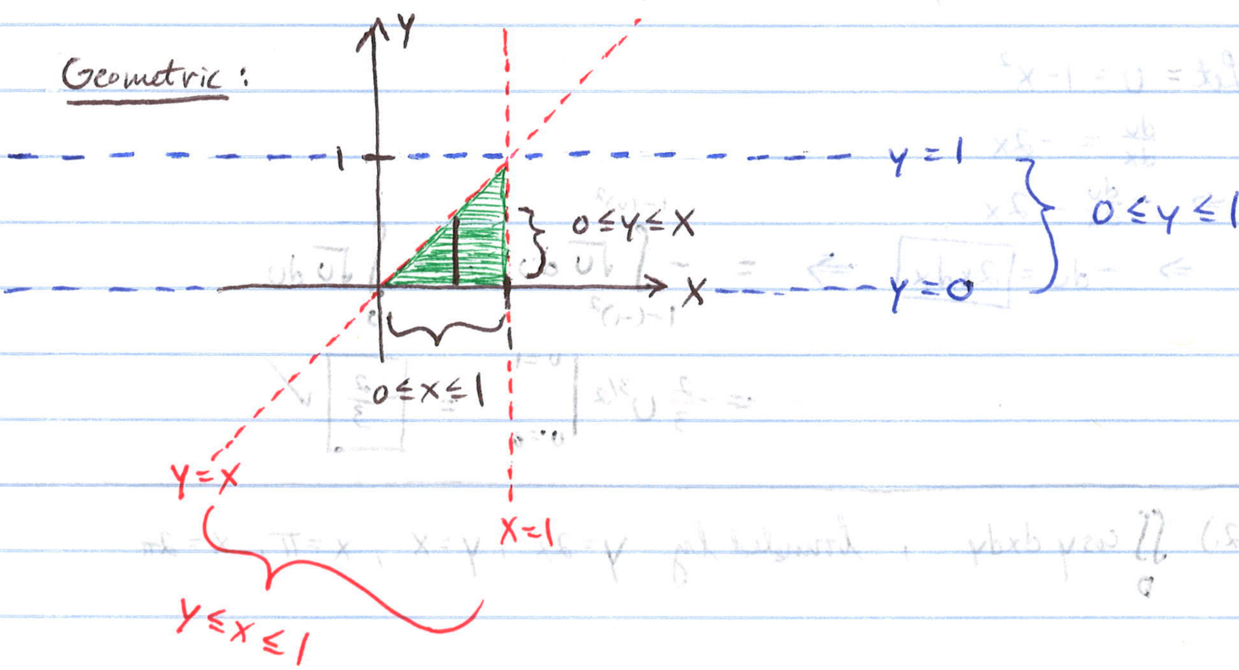
§5.4 2.) $\displaystyle\int_0^1\int_y^1 \sin(x^2)\,dx\,dy$ $\qquad \begin{cases} y \le x \le 1 \\ 0 \le y \le 1 \end{cases}$

Algebraic : $0 \le y \le x \le 1$ (combine both inequalities)

$\Rightarrow \begin{cases} 0 \le y \le x \\ 0 \le x \le 1 \end{cases}$ new system of inequalities

Geometric :



$$\Rightarrow \int_0^1 \int_y^1 \sin(x^2)\,dx\,dy = \int_0^1 \int_0^x \sin(x^2)\,dy\,dx$$

$$= \int_0^1 y\sin(x^2)\Big|_{y=0}^{y=x} dx$$

$$= \int_0^1 \underline{x\sin(x^2)\,dx}$$

Let $u = x^2$

$\frac{du}{dx} = 2x$

$\Rightarrow \frac{1}{2}\frac{du}{dx} = x$

$\Rightarrow \frac{1}{2}du = \boxed{x\,dx}$

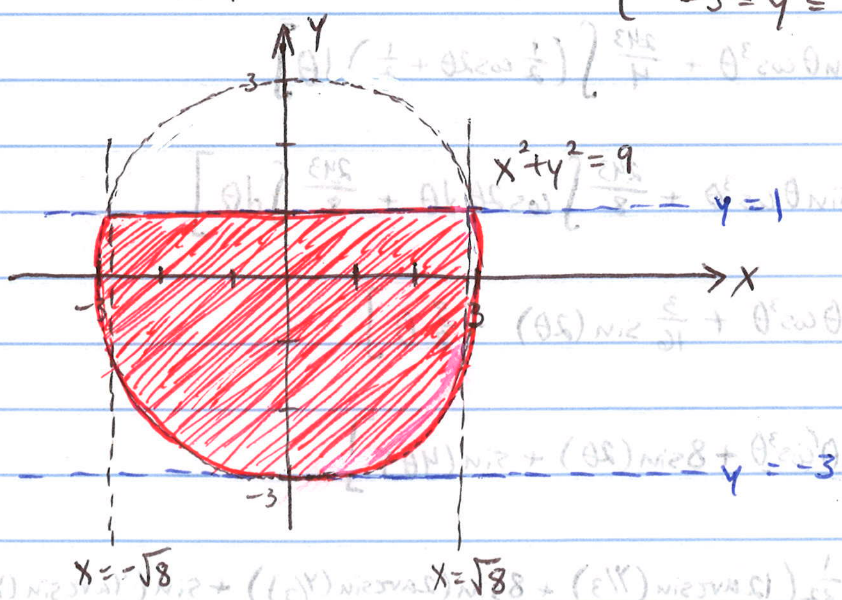
$$\Rightarrow \quad = \int_{(0)^2}^{(1)^2} \frac{1}{2}\sin(u)\,du$$

$$= \int_0^1 \frac{1}{2}\sin(u)\,du$$

$$= -\frac{1}{2}\cos(u)\Big|_{u=0}^{u=1}$$

$$= \boxed{-\frac{1}{2}(\cos(1)-1)} \checkmark$$

4.) b.) $\displaystyle\int_{-3}^{1}\int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} x^2\,dx\,dy$ $\quad \begin{cases} -\sqrt{9-y^2} \le x \le \sqrt{9-y^2} \\ -3 \le y \le 1 \end{cases}$



<u>Without changing</u> order of integration :

$$= \int_{-3}^{1} \frac{1}{3}x^3 \Big|_{x=-\sqrt{9-y^2}}^{x=\sqrt{9-y^2}} \, dy$$

$$= \frac{1}{3}\int_{-3}^{1} (9-y^2)^{3/2} - (-\sqrt{9-y^2}) \quad dy$$

$$= \frac{1}{3}\int_{-3}^{1} 2(9-y^2)^{3/2} dy = \frac{2}{3}\int_{-3}^{1} (9-y^2)\sqrt{9-y^2}\, dy \quad (*)$$

Let $y = 3\sin\theta \longleftrightarrow \theta = \arcsin(y/3)$

- $\frac{dy}{d\theta} = 3\cos\theta \Rightarrow \boxed{dy = 3\cos\theta\, d\theta}$
- $y^2 = 9\sin^2\theta = 9(1-\cos^2\theta) = 9 - 9\cos^2\theta$

$\Rightarrow 9\cos^2\theta = 9 - y^2$

$$\Rightarrow \quad \frac{2}{3}\int (9-y^2)\sqrt{9-y^2}\, dy = \frac{2}{3}\int 9\cos^2\theta \sqrt{9\cos^2\theta}\; 3\cos\theta\, d\theta$$

$$= \frac{2}{3}\int 81\cos^4\theta\, d\theta = \frac{162}{3}\int \cos^4\theta\, d\theta$$

$$= \frac{162}{3}\left[\frac{1}{4}\sin\theta\cos^3\theta + \frac{3}{4}\int \cos^2\theta\, d\theta\right]$$

$$= \frac{162}{3}\left[\frac{1}{4}\sin\theta\cos^3\theta + \frac{3}{4}\int\left(\frac{1}{2}\cos 2\theta + \frac{1}{2}\right)d\theta\right]$$

$$= \frac{162}{3}\left[\frac{1}{4}\sin\theta\cos^3\theta + \frac{3}{8}\int \cos 2\theta\, d\theta + \frac{3}{8}\int d\theta\right]$$

$$= \frac{162}{3}\left[\frac{1}{4}\sin\theta\cos^3\theta + \frac{3}{16}\sin(2\theta) + \frac{3}{8}\theta\right]$$

$$= \frac{162}{3}\left[\frac{1}{32}\left(12\theta + 8\sin(2\theta) + \sin(4\theta)\right)\right]$$

$$= \frac{162}{3}\left[\frac{1}{32}\left(12\arcsin(y/3) + 8\sin(2\arcsin(y/3)) + \sin(4\arcsin(y/3))\right)\right]$$

$$\Rightarrow \int x^2\sqrt{9-x^2}\,dx = \int 9\sin^2\theta\,\sqrt{9\cos^2\theta}\;3\cos\theta\,d\theta$$

$$= \boxed{81\int \sin^2\theta\cos^2\theta\,d\theta}$$

$$= 81\int \sin^2\theta\left(1-\sin^2\theta\right)d\theta$$

$$= 81\int \sin^2\theta - \sin^4\theta\,d\theta$$

$$\vdots$$

$$= \frac{81}{8}\theta + \frac{81}{4}\sin^3\theta\cos\theta - \frac{81}{8}\sin\theta\cos\theta$$

$$= -\frac{9}{8}\sqrt{9-x^2} + \frac{1}{4}x^3\sqrt{9-x^2} + \frac{81}{8}\arcsin\left(\frac{x}{3}\right)$$

$$= \frac{1}{8}\left(2x^3-9x\right)\sqrt{9-x^2} + \frac{81}{8}\arcsin\left(\frac{x}{3}\right)$$

$$\Rightarrow 4\int_{\sqrt{8}}^{3} x^2\sqrt{9-x^2}\,dx = \cancelto{0}{\frac{1}{2}\left(2(3)^3-9(3)\right)\sqrt{9-3^2}} + \frac{81}{2}\cancelto{\pi/2}{\arcsin\left(\frac{3}{3}\right)}$$

$$- \frac{1}{2}\left(2(\sqrt{8})^3-9(\sqrt{8})\right)\cancelto{1}{\sqrt{9-8}} - \frac{81}{2}\arcsin\left(\frac{\sqrt{8}}{3}\right)$$

$$= \frac{81\pi}{4} - \frac{7\sqrt{8}}{2} - \frac{81}{2}\arcsin\left(\frac{\sqrt{8}}{3}\right)$$

$$= \frac{81\pi}{4} - 7\sqrt{2} - \frac{81}{2}\arcsin\left(\frac{2\sqrt{2}}{3}\right)$$

$$\Rightarrow 2\int_{0}^{\sqrt{8}} x^2\sqrt{9-x^2}\,dx = \frac{1}{4}\left(2(\sqrt{8})^3-9\sqrt{8}\right)\cancelto{1}{\sqrt{9-8}} + \frac{81}{4}\arcsin\left(\frac{\sqrt{8}}{3}\right)$$

$$- \cancelto{0}{\frac{1}{4}\left(2(0)^3-9(0)\right)\sqrt{9-0^2}} - \frac{81}{4}\cancelto{0}{\arcsin(0)}$$

$$= \frac{7\sqrt{8}}{4} + \frac{81}{4}\arcsin\left(\frac{\sqrt{8}}{3}\right)$$

$$= \frac{7}{\sqrt{2}} + \frac{81}{4}\arcsin\left(\frac{2\sqrt{2}}{3}\right)$$

$$\Rightarrow (*) = \frac{2}{3}\left(\frac{1}{8}\right)\left(y\sqrt{9-y^2}\,(45-2y^2) + 243\arcsin(y/3) \right)\Big|_{y=-3}^{y=1}$$

$$= \boxed{\frac{43}{3\sqrt{2}} + \frac{81\pi}{8} + \frac{81}{4}\arcsin\left(\frac{1}{3}\right)} \approx \underline{\underline{48.826}}$$

<u>with changing</u> order of integration :

$$= \int_{-3}^{-\sqrt{8}}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} x^2\,dy\,dx + \int_{-\sqrt{8}}^{\sqrt{8}}\int_{-\sqrt{9-x^2}}^{1} x^2\,dy\,dx$$

$$+ \int_{\sqrt{8}}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} x^2\,dy\,dx$$

$$= 2\int_{\sqrt{8}}^{3}\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} x^2\,dy\,dx + 2\int_{0}^{\sqrt{8}}\int_{-\sqrt{9-x^2}}^{1} x^2\,dy\,dx$$

$$= 2\int_{\sqrt{8}}^{3} yx^2\Big|_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}} dx + 2\int_{0}^{\sqrt{8}} yx^2\Big|_{y=-\sqrt{9-x^2}}^{y=1} dx$$

$$= 4\int_{\sqrt{8}}^{3} x^2\sqrt{9-x^2}\,dx + 2\int_{0}^{\sqrt{8}} x^2 + x^2\sqrt{9-x^2}\,dx$$

$$= 4\int_{\sqrt{8}}^{3} x^2\sqrt{9-x^2}\,dx + 2\int_{0}^{\sqrt{8}} x^2\,dx + 2\int_{0}^{\sqrt{8}} x^2\sqrt{9-x^2}\,dx$$

$$= \frac{32\sqrt{2}}{3} + 4\int_{\sqrt{8}}^{3} x^2\sqrt{9-x^2}\,dx + 2\int_{0}^{\sqrt{8}} x^2\sqrt{9-x^2}\,dx \quad (**)$$

Let $x = 3\sin\theta \iff \theta = \arcsin(x/3)$

- $\frac{dx}{d\theta} = 3\cos\theta \Rightarrow \boxed{dx = 3\cos\theta\, d\theta}$
- $x^2 = 9\sin^2\theta = 9(1-\cos^2\theta) \Rightarrow 9 - x^2 = 9\cos^2\theta$

$$\Rightarrow (**) = \frac{32\sqrt{2}}{3} + \frac{81\pi}{4} - 7\sqrt{2} - \frac{81}{2}\arcsin\left(\frac{2\sqrt{2}}{3}\right) + \frac{7}{\sqrt{2}} + \frac{81}{4}\arcsin\left(\frac{2\sqrt{2}}{3}\right)$$

$$= \boxed{\frac{43}{3\sqrt{2}} + \frac{81\pi}{4} - \frac{81}{4}\arcsin\left(\frac{2\sqrt{2}}{3}\right)} \approx 48.826 \checkmark$$

the same answer, different representation

4.) d.) $\displaystyle\int_0^1 \int_{\tan^{-1} y}^{\pi/4} (\sec^5 x)\, dx\, dy$
$\left\{ \begin{array}{l} \tan^{-1} y \le x \le \pi/4 \\ 0 \le y \le 1 \end{array} \right.$

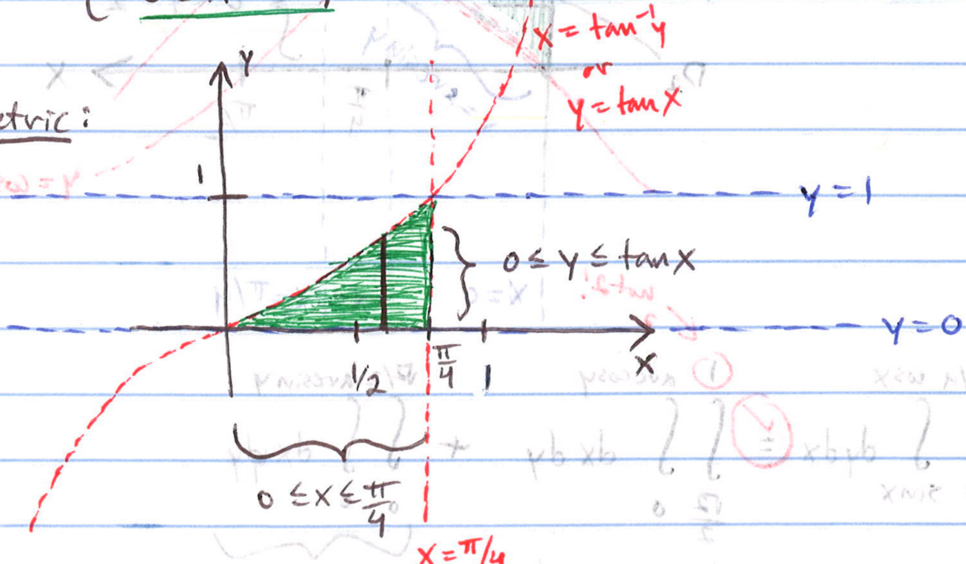
Algebraic: $\tan^{-1} y \le x \le \pi/4$ $\quad 0 \le y \le 1$

$\Rightarrow y \le \tan x \le \tan \pi/4 = 1$ $\quad \tan^{-1}(0) \le \tan^{-1}(y) \le \tan^{-1}(1)$

$\Rightarrow 0 \le \tan^{-1} y \le \frac{\pi}{4}$

$$\Rightarrow \left\{ \begin{array}{l} 0 \le y \le \tan x \\ 0 \le x \le \pi/4 \end{array} \right.$$

Geometric:



$$\Rightarrow \int_0^1 \int_{\tan^{-1} y}^{\pi/4} \sec^5 x\, dx\, dy = \int_0^{\pi/4} \int_0^{\tan x} \sec^5 x\, dy\, dx$$

$$= \int_0^{\pi/4} y\sec^5 x \Big|_{y=0}^{y=\tan x} dx = \int_0^{\pi/4} \tan x \sec^5 x \, dx = \int_0^{\pi/4} \sec^4 x \, \underline{(\sec x \tan x)} \, dx$$

Let $u = \sec x$

$\frac{du}{dx} = \sec x \tan x$

$\Rightarrow du = \boxed{\sec x \tan x \, dx}$

$$\Rightarrow \quad = \int_{\sec(0)}^{\sec(\pi/4)} u^4 \, du$$

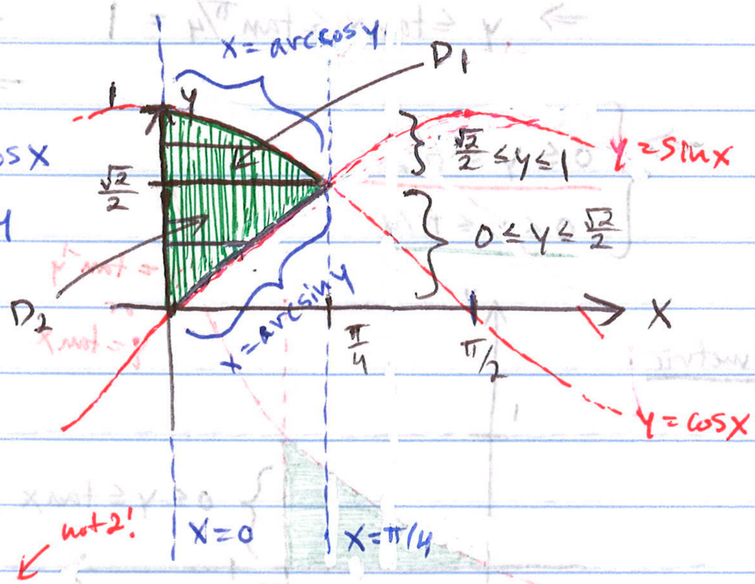
$$= \frac{1}{5} u^5 \Big|_{u=1}^{u=\sqrt{2}}$$

$$= \frac{1}{5}\left((\sqrt{2})^5 - 1\right)$$

$$= \boxed{\frac{1}{5}(4\sqrt{2}-1)} \checkmark$$

6.) a.) $\int_0^{\pi/4} \int_{\sin x}^{\cos x} dy\,dx \overset{?}{=} \int_0^{\sqrt{2}/2} \int_0^{\arcsin y} dx\,dy + \int_{\sqrt{2}/2}^{2} \int_0^{\arccos y} dx\,dy$ FALSE!

$$\begin{cases} \sin x \le y \le \cos x \\ 0 \le x \le \pi/4 \end{cases}$$

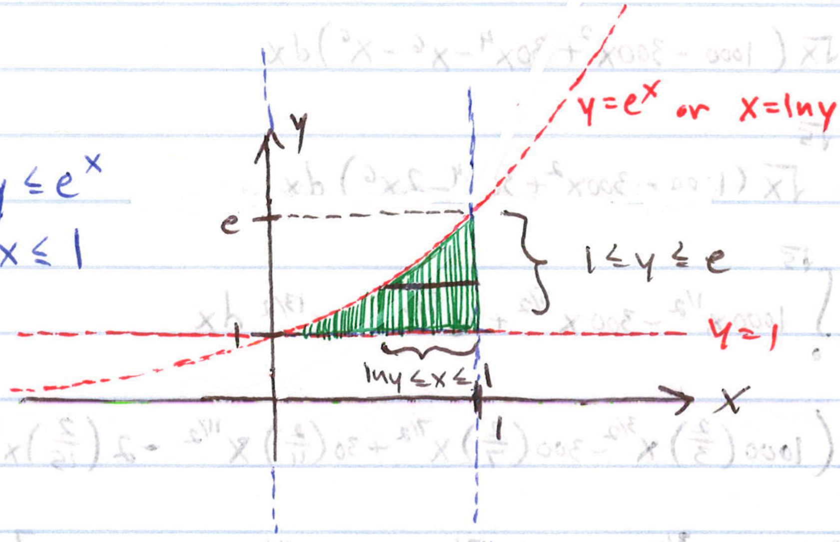


not 2!

Hence $\int_0^{\pi/4} \int_{\sin x}^{\cos x} dy\,dx \overset{\checkmark}{=} \underbrace{\int_{\sqrt{2}/2}^{1} \int_0^{\arccos y} dx\,dy}_{D_1} + \underbrace{\int_0^{\sqrt{2}/2} \int_0^{\arcsin y} dx}_{D_2} \, y$

6.) d.) $\int_0^1\int_1^{e^x} dy\,dx \overset{?}{=} \int_1^e\int_{\ln y}^1 dx\,dy$ TRUE!

$$\begin{cases} 1 \le y \le e^x \\ 0 \le x \le 1 \end{cases}$$



$$\Rightarrow \begin{cases} 1 \le y \le e \\ \ln y \le x \le 1 \end{cases}$$

12.) $f(x,y) = y^2\sqrt{x}$

$$D = \begin{cases} x > 0 \\ y > x^2 \\ y < 10 - x^2 \end{cases} \qquad \begin{cases} x^2 < y < 10 - x^2 \\ 0 < x < \sqrt{5} \end{cases}$$

note: when $x^2 = 10 - x^2$

$\Rightarrow 2x^2 = 10$

$\Rightarrow x^2 = 5$

$\Rightarrow x = \pm\sqrt{5}$

$\Rightarrow \underline{x = \sqrt{5} > 0}$

$$\Rightarrow \iint_D f(x,y)\,dA = \int_0^{\sqrt{5}}\int_{x^2}^{10-x^2} y^2\sqrt{x}\,dy\,dx$$

$$= \int_0^{\sqrt{5}} \frac{1}{3}y^3\sqrt{x}\Big|_{y=x^2}^{y=10-x^2} dx$$

$$= \frac{1}{3}\int_0^{\sqrt{5}} (10-x^2)^3\sqrt{x} - (x^2)^3\sqrt{x}\ dx$$

$$= \frac{1}{3}\int_0^{\sqrt{5}} \sqrt{x}\,(1000 - 300x^2 + 30x^4 - x^6 - x^6)\,dx$$

$$= \frac{1}{3}\int_0^{\sqrt{5}} \sqrt{x}\,(1000 - 300x^2 + 30x^4 - 2x^6)\,dx$$

$$= \frac{1}{3}\int_0^{\sqrt{5}} 1000x^{1/2} - 300x^{5/2} + 30^{9/2} - 2x^{13/2}\ dx$$

$$= \frac{1}{3}\left(1000\left(\tfrac{2}{3}\right)x^{3/2} - 300\left(\tfrac{2}{7}\right)x^{7/2} + 30\left(\tfrac{2}{11}\right)x^{11/2} - 2\left(\tfrac{2}{15}\right)x^{15/2}\right)\Bigg|_0^{\sqrt{5}}$$

$$= \frac{1}{3}\left[\frac{2000}{3}(5)^{3/4} - \frac{600}{7}(5)^{7/4} + \frac{60}{11}(5)^{11/4} - \frac{4}{15}(5)^{15/4}\right]$$

$$= \boxed{\frac{78{,}800}{693}\,5^{3/4}}\ \checkmark$$

18.) Prove $2\int_a^b\int_x^b f(x)f(y)\,dy\,dx = \left(\int_a^b f(x)\,dx\right)^2$

<u>note</u>: $\left(\int_a^b f(x)dx\right)^2 = \left(\int_a^b f(x)\,dx\right)\left(\int_a^b f(x)\,dx\right)$

$$= \left(\int_a^b f(x)\,dx\right)\left(\int_a^b f(y)\,dy\right)$$

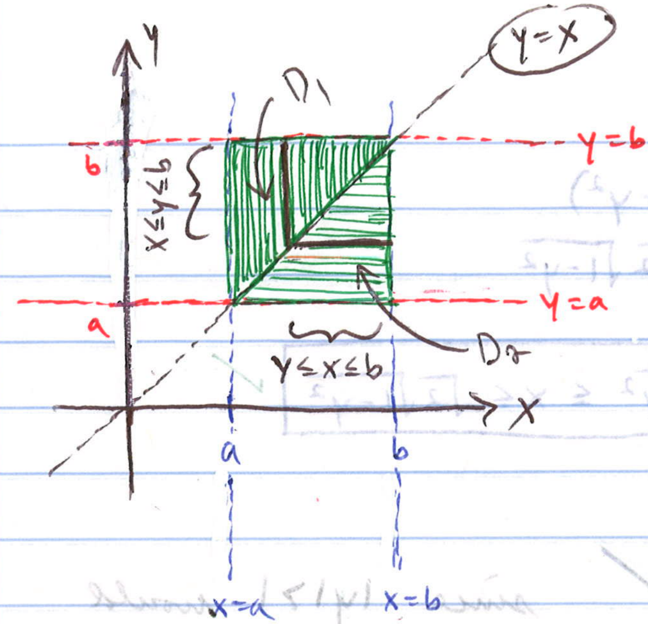
DUMMY VARIABLE

$$= \int_a^b\left(\int_a^b f(y)dy\right)f(x)\,dx$$

$$= \int_a^b\left((f(x))\int_a^b f(y)\,dy\right)dx$$

$$= \int_a^b\left(\int_a^b f(x)f(y)\,dy\right)dx$$

$$= \iint_{[a,b]\times[a,b]} f(x)f(y)\,dx\,dy$$



$$\Rightarrow \iint_{[a,b]\times[a,b]} f(x)f(y)\,dx\,dy = \iint_{D_1} f(x)f(y)\,dx\,dy + \iint_{D_2} f(x)f(y)\,dx\,dy$$

$$= \int_a^b\int_x^b f(x)f(y)\,dy\,dx + \int_a^b\int_y^b f(x)f(y)\,\underline{dx\,dy}$$

$$= 2\int_a^b\int_x^b f(x)f(y)\,\underline{dy\,dx}$$

DUMMY VARIABLES

two copies

$$\Rightarrow \left(\int_a^b f(x)\,dx\right)^2 = 2\int_a^b\int_x^b f(x)f(y)\,dy\,dx \quad \checkmark \ \blacksquare$$

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§ 5.5

12.) $\begin{cases} x^2+2y^2 = 2 \\ \underline{z = 0} \\ x+y+2z = 2 \end{cases} \Rightarrow 2z = 2-(x+y)$

$$\Rightarrow \underline{z = 1-\tfrac{1}{2}(x+y)}$$

$$\Rightarrow \boxed{0 \le z \le 1-\tfrac{1}{2}(x+y)} \checkmark$$

$$x^2+2y^2=2 \Rightarrow x^2=2(1-y^2)$$

(an elipse) $\Rightarrow x=\pm\sqrt{2}\sqrt{1-y^2}$

Hence $\boxed{-\sqrt{2}\sqrt{1-y^2}\le x\le\sqrt{2}\sqrt{1-y^2}}$ ✓

Finally, $\boxed{-1\le y\le 1}$ since $|y|>1$ would mean $1-y^2<0$, and no values on $x^2+2y^2=2$ would be possible.

$$\Rightarrow \int_{-1}^{1}\int_{-\sqrt{2}\sqrt{1-y^2}}^{\sqrt{2}\sqrt{1-y^2}}\int_{0}^{1-\frac{1}{2}(x+y)} dz\,dx\,dy$$

$$=\int_{-1}^{1}\int_{-\sqrt{2}\sqrt{1-y^2}}^{\sqrt{2}\sqrt{1-y^2}} z\Big|_{z=0}^{z=1-\frac{1}{2}(x+y)} dx\,dy$$

$$=\int_{-1}^{1}\int_{-\sqrt{2}\sqrt{1-y^2}}^{\sqrt{2}\sqrt{1-y^2}} 1-\tfrac{1}{2}(x+y)\,dx\,dy$$

$$=\int_{-1}^{1} x-\tfrac{1}{4}x^2-\tfrac{1}{2}xy\Big|_{x=-\sqrt{2}\sqrt{1-y^2}}^{x=\sqrt{2}\sqrt{1-y^2}} dy$$

$$=\int_{-1}^{1}\left[\sqrt{2}\sqrt{1-y^2}-\tfrac{1}{4}\cancel{(2(1-y^2))}-\tfrac{1}{2}y(\sqrt{2}\sqrt{1-y^2})\right]-\left[-\sqrt{2}\sqrt{1-y^2}-\tfrac{1}{4}\cancel{(2(1-y^2))}+\tfrac{1}{2}y\sqrt{2}\sqrt{1-y^2}\right]dy$$

$$=\int_{-1}^{1} 2\sqrt{2}\sqrt{1-y^2}-\sqrt{2}\,y\sqrt{1-y^2}\,dy$$

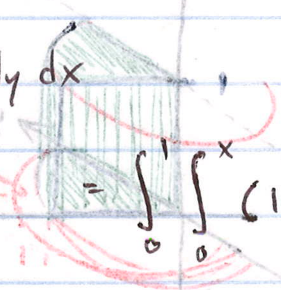
$$= 2\sqrt{2}\underbrace{\int_{-1}^{1}\sqrt{1-y^2}\,dy}_{\substack{\text{area of half circle} \\ \text{w/ radius 1} \\ = \frac{\pi}{2}}} - \sqrt{2}\cancel{\int_{-1}^{1} y\underbrace{\sqrt{1-y^2}}_{\text{odd function, so integral is zero}}dy}^{0}$$

$$= 2\sqrt{2}\left(\frac{\pi}{2}\right) = \boxed{\sqrt{2}\pi} \checkmark$$

16.) $\displaystyle\int_0^1\int_0^x\int_0^y (y+xz)\,dz\,dy\,dx$

$$= \int_0^1\int_0^x zy + \tfrac{1}{2}xz^2\Big|_{z=0}^{z=y} dy\,dx$$

$$= \int_0^1\int_0^x y^2 + \tfrac{1}{2}xy^2\,dy\,dx = \int_0^1\int_0^x (1+\tfrac{1}{2}x)y^2\,dy\,dx$$



$$= \int_0^1 \frac{1}{3}y^3(1+\tfrac{1}{2}x)\Big|_{y=0}^{y=x} dx$$

$$= \frac{1}{3}\int_0^1 x^3(1+\tfrac{1}{2}x)\,dx = \frac{1}{3}\int_0^1 x^3 + \tfrac{1}{2}x^4\,dx$$

$$= \frac{1}{3}\left[\frac{1}{4}x^4 + \frac{1}{10}x^5\Big|_0^1\right]$$

$$= \frac{1}{3}\left[\frac{1}{4} + \frac{1}{10}\right]$$

$$= \boxed{\frac{7}{60}} \checkmark$$

18.) $\displaystyle\iiint_W z\,dx\,dy\,dz$

Algebraically:

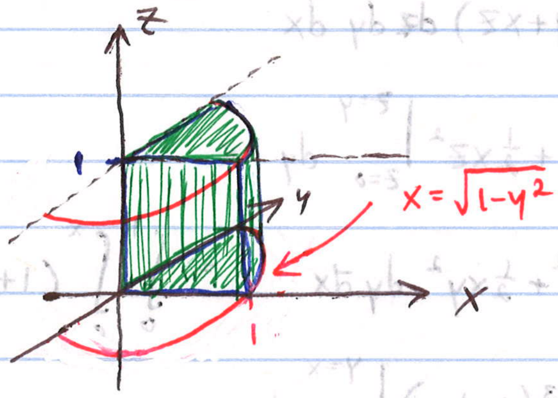
$$\begin{cases} x=0 \\ y=0 \\ \boxed{z=0} \\ x^2+y^2=1, \quad x\geq 0, \quad y\geq 0 \\ \boxed{z=1} \end{cases} \qquad \boxed{0\leq z\leq 1}$$

$$x^2+y^2=1 \Rightarrow x^2 = 1-y^2$$
$$\Rightarrow x = \pm\sqrt{1-y^2} \text{, but } x \geq 0$$

$$\Rightarrow \boxed{0 \leq x \leq \sqrt{1-y^2}}$$

$y \geq 0$, and so $\boxed{0 \leq y \leq 1}$ or else $\sqrt{1-y^2}$ will not be defined

Geometrically:



quarter-cylinder (first quadrant) of height one

$$= \int_0^1\int_0^1\int_0^{\sqrt{1-y^2}} z\,dx\,dy\,dz = \int_0^1\int_0^1 xz\Big|_{x=0}^{x=\sqrt{1-y^2}} dy\,dz$$
$$= \int_0^1\int_0^1 z\sqrt{1-y^2}\,dy\,dz$$
$$= \int_0^1 z\Big(\underbrace{\int_0^1\sqrt{1-y^2}\,dy}_{\text{area of quarter-circle with radius 1} = \frac{\pi}{4}}\Big)\,dz$$

$$= \int_0^1 \frac{\pi}{4}z\,dz = \frac{\pi}{8}z^2\Big|_0^1$$
$$= \boxed{\frac{\pi}{8}}$$