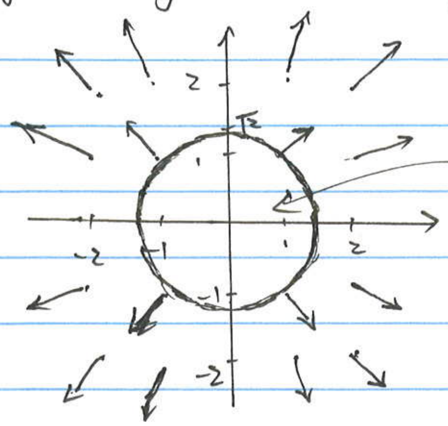
Homework 7. Cong Ma.

4.3 Vector Fields.

22. 24. 26.

22 $f(x,y) = x^2+y^2$. then $\nabla f = (2x, 2y)$



By observation they are perpendicular to the level sets.

24. let $c(t)$ be a flow line of a gradient field $F = -\nabla V$.
prove that $V(c(t))$ is a decreasing function of t.

By Definition $c'(t) = -\nabla V(c(t))$

and $(V(c(t)))'$ over t gives us that.

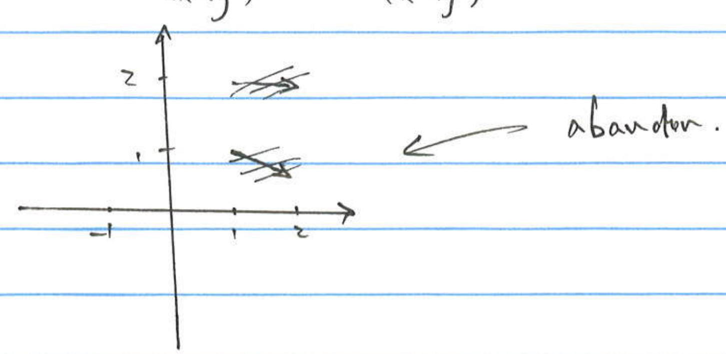
$$(V(c(t)))' = \nabla V(c(t)) \cdot c'(t)$$

$$= -[\nabla V(c(t))]^2 \text{ is non positive}$$

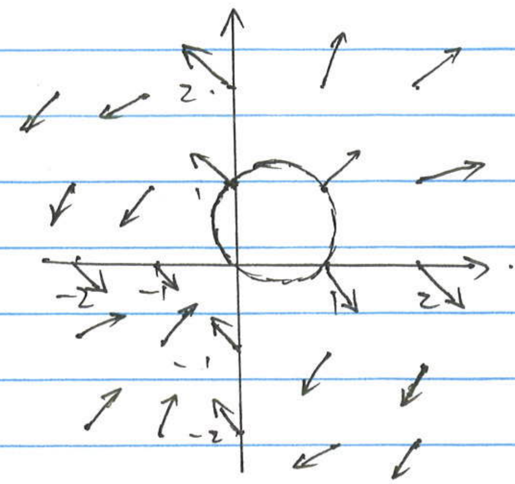
$\Rightarrow$ $V(c(t))$ is decreasing.

26. $V(x,y) = (x+y)/(x^2+y^2)$

$$-\nabla V = \left(\frac{-y^2+x^2+2xy}{(x^2+y^2)^2}, \frac{-x^2+y^2+2xy}{(x^2+y^2)^2} \right).$$



26.



44. Divergence and Curl.

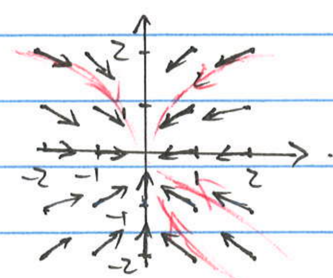
4. $V(x,y,z) = x^2 i + (x+y)^2 j + (x+y+z)^2 k$.

$\operatorname{div} V = 2x + 2(x+y)\cdot 1 + 2(x+y+z)\cdot 1$

$= 6x + 4y + 2z$.

8. $F(x,y) = -3xi - yj$

$F(x,y) = (-3x, -y)$



$\nabla \cdot F = -3 - 1 = -4$ so divergence is negative

that tells us that the gas or fluid is compressing

just like the sketch.

16. $F(x,y,z) = \dfrac{yz\,i - xz\,j + xy\,k}{x^2+y^2+z^2}$. Let's say $x^2+y^2+z^2 = *$.

$$\frac{\partial F_3}{\partial y} = \frac{x(x^2+y^2+z^2) - 2y(xy)}{(x^2+y^2+z^2)^2}$$

$$\frac{\partial F_2}{\partial z} = \frac{-x \cdot * + xz(2z)}{*^2}$$

$$\frac{\partial F_3}{\partial y} - \frac{\partial F_2}{\partial z} = \frac{(x+x)(*) - 2y^2x - 2xz^2}{*^2} = A = \frac{2x^3}{*^2}$$

$$\frac{\partial F_1}{\partial z} = \frac{y(*) - yz(2z)}{*^2}$$

$$\frac{\partial F_3}{\partial x} = \frac{y(*) - xy(2x)}{*^2}$$

$$\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial x} = \frac{2x^2y - 2yz^2}{*^2} = B.$$

$$\frac{\partial F_2}{\partial x} = \frac{-z \cdot * + xz(2x)}{*^2}$$

$$\frac{\partial F_1}{\partial y} = \frac{z \cdot * - yz(2y)}{*^2}$$

$$\frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} = \frac{-2z \cdot * + 2x^2z + 2y^2z}{*^2} = C = \frac{-2z^3}{*^2}$$

$$\nabla \times F = Ai + Bj + Ck$$

22. (a) (13) $F(x,y,z) = xi + yj + zk$.

$$\frac{\partial F}{\partial x} = x \quad \frac{\partial F}{\partial y} = y \quad \frac{\partial F}{\partial z} = z.$$

then $f = \frac{1}{2}x^2 + \frac{1}{2}y^2 + \frac{1}{2}z^2$ fits.

(14). $f = xyz$ fits.

(15) $f = \cancel{y}\, x^3 + \frac{4}{3}y^3 + \frac{5}{3}z^3$ fits.

(16). its curl of gradients is not zero.

then it's not a gradient fields of 16.

(b). (9) $\text{div} \cdot F(x,y) \neq 0$ then it can't.

(10). $\text{div} \cdot F(x,y) = 0$ it works.

(11) $\text{div}\, F(x,y) = y\cos(xy) + \sin(x^2y)x^2 \neq 0$ then it can't.

(12) $\text{div}\, F(x,y) = e^y \cancel{\oplus} \left(\frac{x+y-y}{(x+y)^2}\right) \neq 0$ then it can't.

24. (a) curl (grad F) meaningful vector function

(b) grad (curl f) meaningless.

(c) div (grad F) meaningful scalar function.

(d) grad (div F) meaningless. (f) div (curl F) meaningless.

(e) curl (div F) meaningless.

32 $\nabla f = (2xy^2,\ 2x^2y + 2yz^2,\ 2zy^2)$

$$\nabla \times (\nabla f) = (4zy - 4zy)i + (0-0)j + (4xy - 4xy)j$$
$$= 0.$$

40. (a) $\nabla \times (F) = (0-0)i + (0-0)j + (3x^2 - 3x^2)k = 0.$

$F = 3x^2y\,i + (x^3+y^3)j$

(b) $\frac{\partial F}{\partial x} = 3x^2y \quad F = x^3y + C$

$\frac{\partial F}{\partial y} = x^3 + y^3 \quad F = x^3y + \frac{1}{4}y^4 + C'$

$\Rightarrow$ there is one $f = x^3y + \frac{1}{4}y^4 + C''$

5.1 Introduction.

2 (a) $\int_0^1\int_0^1 1-x^3+xy\,dy\,dx$

$= \int_0^1 y-x^3y+\frac{1}{2}xy^2\Big|_0^1\,dx$

$= \int_0^1 1-x^3+\frac{1}{2}x\,dx$

$= x-\frac{1}{4}x^4+x^2\Big|_0^1$

$= 1-\frac{1}{4}+1 = \frac{7}{4}$

(b) $\int_0^{\frac{\pi}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\sin y\,dx\,dy$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \cos x\sin y\,dy\,dx$

$= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos x\,dx.$

$= \sin x\Big|_{-\frac{\pi}{2}}^{\frac{\pi}{2}}$

$= 1+1 = 2.$

(c) $\int_1^2\int_2^4 (\frac{x}{y}+\frac{y}{x})\,dx\,dy.$

$= \int_2^4\int_1^2 \frac{x}{y}+\frac{y}{x}\,dy\,dx$

$= \int_2^4 (x\ln|y| + \frac{y^2}{2x}\Big|_1^2)dx$

$= \int_2^4 x\cdot\ln 2+\frac{4}{2x}-\frac{1}{2x}\,dx$

$= \int_2^4 x\cdot\ln 2+\frac{3}{2x}\,dx$

$= \frac{1}{2}x^2\ln 2+\frac{3}{2}\cdot\ln|x|\Big|_2^4$

$= 8\ln 2+\frac{3}{2}\ln \cancel{4}^{2} - 2\ln 2-\frac{3}{2}\ln 2$

$= 9\ln 2-\frac{3}{2}\ln 2$

$= \frac{15}{2}\ln 2$

(d) $\int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} \tan x \sec^2 y \, dx\, dy.$

$$= \int_0^{\frac{\pi}{4}} \int_0^{\frac{\pi}{4}} \tan x \sec^2 y \, dy\, dx$$

$$= \int_0^{\frac{\pi}{4}} \tan x \tan y \Big|_0^{\frac{\pi}{4}} dx$$

$$= \int_0^{\frac{\pi}{4}} \tan x \, dx$$

$$= -\log|\cos x| \Big|_0^{\frac{\pi}{4}}$$

$$= -\log \frac{\sqrt{2}}{2}$$

12 $\int_1^3 \int_1^2 \frac{xy}{(x^2+y^2)^{3/2}} \, dx\, dy.$

$x^2+y^2 = u$

$\frac{du}{dx} = 2x$

$\frac{du}{2} = x\,dx$

$$= \int_1^3 \int_1^2 \frac{y}{2} \cdot \frac{du}{u^{3/2}} \, dy$$

$$= \int_1^3 \frac{-y}{\sqrt{x^2+y^2}} \Big|_1^2 \, dy.$$

$y^2+c = u$

$\frac{du}{dy} = 2y$

$\frac{du}{2} = y\,dy$

$$= \int_1^3 \frac{-y}{\sqrt{y^2+4}} + \frac{y}{\sqrt{y^2+1}} \, dy$$

$$= -\sqrt{y^2+4} + \sqrt{y^2+1} \Big|_1^3$$

$$= -\sqrt{13} + \sqrt{10} + \sqrt{5} - \sqrt{2}.$$