HW #5

§3.2

2.) Suppose $L: \mathbb{R}^2 \to \mathbb{R}$ is linear, so that L has the form $L(x,y) = ax + by$.

a.) Find the first-order taylor approximation for L.

$L_x = a$, $L_y = b$, hence if $g(x,y)$ is the first-order approximation, then:

$$\begin{aligned} g(x,y) &= L(0,0) + L_x(0,0)\,x + L_y(0,0)\,y \\ &= 0 + a(x) + b(y) \\ &= ax + by \\ &= L(x,y) \checkmark \end{aligned}$$

Hence the first-order approximation to a linear function is itself.

b.) Find the second-order taylor approximation for L.

$L_{xx} = L_{xy} = L_{yy} = 0$, and so the second order approximation is:

$$\begin{aligned} g(x,y) &= L(0,0) + L_x(0,0)\,x + L_y(0,0)\,y + \tfrac{1}{2}L_{xx}(0,0)\,x^2 + \tfrac{1}{2}L_{yy}(0,0)\,y^2 \\ &\quad + L_{xy}(0,0)\,xy \\ &= 0 + ax + by + \tfrac{1}{2}(0)\,x^2 + \tfrac{1}{2}(0)\,y^2 + (0)\,xy \\ &= ax + by \\ &= L(x,y) \checkmark \end{aligned}$$

c.) What will higher-order approximations look like?

Since all partial-derivatives of $L(x,y)$ are zero for degree 2 or higher, all further approximations will still be equal to $L(x,y)$ itself ✓

10.) Let $f(x,y) = x\cos(\pi y) - y\sin(\pi x)$. Find the second-order Taylor approximation for f at the point $(1,2)$.

- $f(1,2) = 1\cos(2\pi) - 2\sin(\pi) = 1(1) - 2(0) = 1$

- $f_x(x,y) = \cos(\pi y) - \pi y\cos(\pi x)$, $f_x(1,2) = \cos(2\pi) - 2\pi\cos(\pi) = 1+2\pi$
- $f_y(x,y) = -\pi x\sin(\pi y) - \sin(\pi x)$, $f_y(1,2) = -\pi\sin(2\pi) - \sin(\pi) = 0$

- $f_{xx}(x,y) = \pi^2 y\sin(\pi x)$, $f_{xx}(1,2) = 2\pi^2\sin(\pi) = 0$
- $f_{yy}(x,y) = -\pi^2 x\cos(\pi y)$, $f_{yy}(1,2) = -\pi^2\cos(2\pi) = -\pi^2$
- $f_{xy}(x,y) = -\pi\sin(\pi y) - \pi\cos(\pi x)$, $f_{xy}(1,2) = -\pi\sin(2\pi) - \pi\cos(\pi) = \pi$

Hence the second-order approximation is

$$f(1,2) + f_x(1,2)(x-1) + f_y(1,2)(y-2) + \tfrac{1}{2}f_{xx}(1,2)(x-1)^2 + f_{xy}(1,2)(x-1)(y-2) + \tfrac{1}{2}f_{yy}(1,2)(y-2)^2$$

$$= 1 + (1+2\pi)(x-1) + \pi(x-1)(y-2) - \tfrac{1}{2}\pi^2(y-2)^2$$ ✓

3.3

6.) $f(x,y) = x^2 - 3xy + 5x - 2y + 6y^2 + 8$

$\nabla f(x,y) = (2x - 3y + 5, -3x - 2 + 12y) = (0,0)$

$$\Rightarrow \begin{cases} 2x - 3y + 5 = 0 \\ -3x - 2 + 12y = 0 \end{cases} \Rightarrow \begin{cases} (2x - 3y = -5) \cdot 4 \\ + \; -3x + 12y = 2 \end{cases}$$

$$5x + 0y = -18$$

$$\Rightarrow x = -18/5$$

$$\Rightarrow (2(-18/5) - 3y + 5 = 0) \cdot 5$$

$$\Rightarrow -15y = -25 + 36 = 11$$

$$\Rightarrow y = -11/15$$

Theorem 6: $\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}(2x - 3y + 5) = 2 > 0$ ✓ (ii)

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}(-3x - 2 + 12y) = 12$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}(-3x - 2 + 12y) = -3$$

$\Rightarrow D = 2(12) - (-3)^2 = 24 - 9 = 15 > 0$ ✓ (iii)

Hence there is a local minimum at $(\frac{-18}{5}, -11/15)$ ✓

14.) $f(x,y) = \log(2 + \sin xy)$

$\nabla f(x,y) = \left(\frac{1}{2 + \sin xy} \cdot y\cos xy, \frac{1}{2 + \sin xy} \cdot x\cos xy\right) = (0,0)$

$$\Rightarrow \begin{cases} y\cos xy = 0 \\ x\sin xy = 0 \end{cases}$$

Hence, either $y=0$ or $\cos xy = 0 \Rightarrow xy = n\pi + \frac{1}{2} \neq 0$

- suppose $y=0 \Rightarrow \cos(xy) = \cos(0) \neq 0$

  so $x\cos(xy) = 0 \Rightarrow x = 0$

  Hence $\boxed{(0,0) \text{ is a critical point}}$

- suppose $\cos xy = 0 \Rightarrow xy = n\pi \pm \frac{\pi}{2}$, n any integer

  $\Rightarrow y = \frac{1}{x}(n\pi \pm \frac{\pi}{2}) = \frac{\pi}{x}(n+\frac{1}{2})$

  Hence $\boxed{(x, \frac{\pi}{x}(n+\frac{1}{2})) \text{ is a critical}}$ point for all x and all integers n.

- $$\frac{\partial^2 f}{\partial x^2} = \frac{-y^2\cos^2(xy)}{(2+\sin xy)^2} + \frac{-y^2 \sin xy}{(2+\sin xy)} = 0 \text{ @ } (0,0) \checkmark$$

$$= \frac{-(\frac{\pi}{x}(n+\frac{1}{2}))^2(-1)^n}{(2+(-1)^n)} = \begin{cases} >0 & \text{for } n \text{ odd} \\ <0 & \text{for } n \text{ even} \end{cases} \text{ @ } (x, \frac{\pi}{x}(n+\frac{1}{2})) \checkmark$$

- $$\frac{\partial^2 f}{\partial y^2} = \frac{-x^2\cos^2(xy)}{(2+\sin xy)^2} + \frac{-x^2\sin(xy)}{2+\sin xy} = 0 \text{ @ } (0,0)$$

$$= \frac{-x^2(-1)^n}{2+(-1)^n} = \begin{cases} >0 & \text{for } n \text{ odd} \\ <0 & \text{for } n \text{ even} \end{cases} \text{ @ } (x, \frac{\pi}{x}(n+\frac{1}{2})) \checkmark$$

- $$\frac{\partial^2 f}{\partial x^2} = \frac{-xy\cos^2 xy}{(2+\sin xy)^2} + \frac{\cos xy}{2+\sin xy} + \frac{-xy\sin xy}{2+\sin xy} = \frac{1}{2} \text{ @ } (0,0)$$

$$= \frac{-\pi(n+1/2)(-1)^n}{2+(-1)^n} \text{ @ } (x, \frac{\pi}{x}(n+\frac{1}{2})) \checkmark$$

• Hence, at $(0,0)$, we have $D = 0(0) - (\frac{1}{2})^2 < 0$
and so there is a saddle point at $(0,0)$ ✓

• at $(x, \frac{\pi}{x}(n+\frac{1}{2}))$, we have

$$D = \left(\frac{-(\frac{\pi}{x}(n+\frac{1}{2}))^2(-1)^n}{(2+(-1)^n)}\right)\left(\frac{-x^2(-1)^n}{2+(-1)^n}\right) - \left(\frac{-\pi(n+\frac{1}{2})(-1)^n}{2+(-1)^n}\right)^2$$

$$= \frac{\pi^2(n+\frac{1}{2})^2 - (\pi^2(n+\frac{1}{2})^2)}{(2+(-1)^n)^2} = 0 \quad \underline{\text{DEGENERATE}} \ ☹$$

• consider then $xy = \pi(n+\frac{1}{2})$
$(x,y) = (x, \frac{\pi}{x}(n+\frac{1}{2})) \Rightarrow \sin(xy) = (-1)^n = \begin{cases} -1 & , n \text{ odd} \\ 1 & , n \text{ even} \end{cases}$

$$\Rightarrow 2+\sin xy = \begin{cases} 1, & n \text{ odd} \\ 3, & n \text{ even} \end{cases}$$

since log is an increasing function, and $1 \le 2+\sin xy \le 3$,
then $(x, \frac{\pi}{x}(n+\frac{1}{2}))$ must be $\begin{cases} \text{local min}, & n \text{ odd} \\ \text{local max}, & n \text{ even} \end{cases}$ ✓

28.) If a point (x,y,z) is in the plane $2x-y+2z=20$
$\Rightarrow y = 2x+2z-20$
$\Rightarrow x^2+y^2+z^2 = x^2+(2x+2z-20)^2+z^2$
$= x^2+4x^2+4z^2+400+8xz-80x-80z+z^2$
$= 5x^2+5z^2+8xz-80x-80z+400 = f(x,y)$

$$\nabla f(x,y) = (10x + 8z - 80,\ 10z + 8x - 80) = (0,0)$$

$$\Rightarrow \begin{cases} 10x + 8z = 80 \\ 10z + 8x = 80 \end{cases} \Rightarrow \begin{cases} (5x + 4z = 40)\cdot(-4) \\ (4x + 5z = 40)\cdot(5) \end{cases}$$

$$0x + 9z = 40$$

$$\Rightarrow z = 40/9$$

$$\Rightarrow 10x + 8\left(\tfrac{40}{9}\right) = 80$$

$$\Rightarrow 10x = \tfrac{400}{9}$$

$$\Rightarrow x = \tfrac{40}{9} \quad \Rightarrow y = 2(40/9) + 2(40/9) - 20$$

$$\Rightarrow y = \tfrac{-20}{9}$$

Hence $\boxed{(x,y,z) = \frac{1}{9}(40,-20,40)}$ is the nearest point to the origin on the plane

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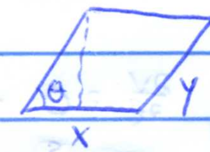
30.) Given a rectangular parallelepiped with base



we have

$$V = xyz\sin\theta$$

$$S = 2xy\sin\theta + 2xz + 2yz$$

Let $\alpha = \sin\theta$. Note, $x,y,z \neq 0$ and $0 < \theta \leq \pi/2$, so $\alpha \neq 0$.

$$\Rightarrow S = 2x[y\alpha + z] + 2yz$$

$$\Rightarrow x = \left(\tfrac{S}{2} - yz\right)/(y\alpha + z)$$

$$\Rightarrow V = \left(\tfrac{S}{2} - yz\right)yz\alpha\,(y\alpha + z)^{-1} = \left(\tfrac{S}{2}yz - y^2z^2\right)\alpha\,(y\alpha + z)^{-1}$$

$$\frac{\partial V}{\partial y} = \left(\tfrac{S}{2}z - 2z^2y\right)\alpha(y\alpha + z)^{-1} - \left(\tfrac{S}{2}yz - y^2z^2\right)\alpha^2(y\alpha + z)^{-2}$$

$$= \left[\left[\tfrac{S}{2}z - 2z^2y\right](y\alpha + z)\alpha - \left(\tfrac{S}{2}yz - y^2z^2\right)\alpha^2\right](y\alpha + z)^{-2}$$

Similarly,

$$\frac{\partial V}{\partial z} = \left(\tfrac{5}{2}y - 2y^2z\right)\alpha(y\alpha+z)^{-1} - \left(\tfrac{5}{2}yz - y^2z^2\right)\alpha(y\alpha+z)^{-2}$$

$$= \alpha\left[\left(\tfrac{5}{2}y - 2y^2z\right)(y\alpha+z) - \left(\tfrac{5}{2}yz - y^2z^2\right)\right](y\alpha+z)^{-2}$$

$$\frac{\partial V}{\partial y} = 0 \Rightarrow \left(\tfrac{5}{2}z - 2z^2y\right)(y\alpha+z)\alpha - \left(\tfrac{5}{2}yz - y^2z^2\right)\alpha^2 = 0$$

$$\Rightarrow \left[\left(\tfrac{5}{2} - 2zy\right)(y\alpha+z) - \left(\tfrac{5}{2}y - y^2z\right)\alpha\right]\alpha z = 0$$

$\alpha \neq 0, z \neq 0$ $$\Rightarrow \left(\tfrac{5}{2} - 2zy\right)(y\alpha+z) - \left(\tfrac{5}{2} - yz\right)y\alpha = 0$$

$$\Rightarrow \cancel{\tfrac{5}{2}\alpha y} - \cancel{2}\alpha zy^2 + \tfrac{5}{2}z - 2yz^2 - \left[\cancel{\tfrac{5}{2}\alpha y} - \cancel{\alpha y^2 z}\right] = 0$$

$$\Rightarrow \tfrac{5}{2}z - 2yz^2 - \alpha zy^2 = 0$$

$$\Rightarrow \tfrac{5}{2} - 2yz - \alpha y^2 = 0$$

$$\Rightarrow \alpha y^2 + 2zy - 5/2 = 0 \checkmark \Rightarrow \boxed{\alpha y^2 + 2yz = 5/2}$$

Similarly $$\frac{\partial V}{\partial z} = 0 \Rightarrow \left(\tfrac{5}{2}\cancel{y} - 2y^{\cancel{2}}z\right)(y\alpha+z) - \left(\tfrac{5}{2}\cancel{y}z - y^{\cancel{2}}z^2\right) = 0$$

$$\Rightarrow \left(\tfrac{5}{2} - 2yz\right)(y\alpha+z) - \left(\tfrac{5}{2}z - yz^2\right) = 0$$

$$\Rightarrow \tfrac{5}{2}\alpha y - 2\alpha y^2 z + \cancel{\tfrac{5}{2}z} - \cancel{2}yz^2 - \cancel{\tfrac{5}{2}z} + \cancel{yz^2} = 0$$

$$\Rightarrow \tfrac{5}{2}\alpha \cancel{y} - 2\alpha y^{\cancel{2}}z - \cancel{y}z^2 = 0$$

$$\Rightarrow \tfrac{5}{2}\alpha - 2\alpha yz - z^2 = 0$$

$$\Rightarrow z^2 + 2\alpha yz - \tfrac{5}{2}\alpha = 0 \checkmark \Rightarrow \boxed{\tfrac{1}{\alpha}z^2 + 2yz = 5/2}$$

$$\Rightarrow \alpha y^2 + \cancel{2yz} = \tfrac{1}{\alpha}z^2 + \cancel{2yz} \Rightarrow \alpha^2y^2 = z^2$$

$$\Rightarrow z = \sqrt{\alpha^2y^2} = \alpha y$$, since $x, y, z, \alpha > 0$

Similarly, if you solve for Z, to get
$Z = (5/2 - xy\alpha)/(x+y)$, and set $\nabla V(x,y) = (0,0)$

you arrive at $\begin{cases} \alpha x^2 + 2xy\alpha = 5/2 \\ \alpha y^2 + 2xy\alpha = 5/2 \end{cases}$

$$\Rightarrow x = y$$

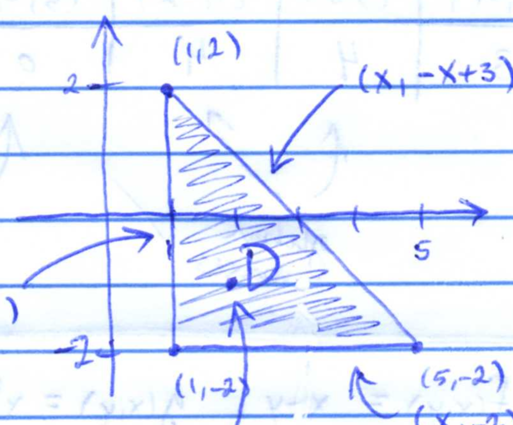
Hence, the max value is when $x = y = \frac{1}{\alpha} z$

$$\Rightarrow V(x,y,z) = xyz\alpha$$
$$= \frac{1}{\alpha} z^3$$

which is largest when $\alpha = \sin\theta = 1$
$\iff \theta = \pi/2$

ie. $x = y = z$ and $\theta = \pi/2$
so a cube!

44.) $f(x,y) = 1 + xy + x - 2y$



- $\nabla f(x,y) = (y+1, x-2) = (0,0)$

$\Rightarrow \begin{cases} y+1=0 \\ x-2=0 \end{cases} \Rightarrow \begin{cases} y=-1 \\ x=2 \end{cases}$ inside D

$(x,y) = (2,-1)$

- parametrize the three sides:

① $f(x,-2) = 1+(-2)x + x - 2(-2)$
$= 5 - x$
$\Rightarrow \frac{d}{dx} f(x,-2) = -1 \neq 0$ ✓

② $f(1,y) = 1 + (1)y + (1) - 2y$
$= 2 - y$
$\Rightarrow \frac{d}{dy} f(1,y) = -1 \neq 0$ ✓

③ $f(x,-x+3) = 1 + x(-x+3) + x - 2(-x+3)$
$= -x^2 + 6x - 5$
$\Rightarrow \frac{d}{dx} f(x,-x+3) = -2x + 6 = 0$
$\Rightarrow x = 3$ Hence $(3,0)$ is a candidate ✓

- Also check any corners, $(1,-2)$, $(5,-2)$, $(1,2)$ ✓

(x,y)	$(2,-1)$	$(3,0)$	$(1,-2)$	$(5,-2)$	$(1,2)$
$f(x,y)$	3	4	4	0	0

global maxes ✓ global mins ✓

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§3.4 4.) $f(x,y) = x - y$, $g(x,y) = x^2 - y^2 = 2$
$\nabla f(x,y) = (1,-1)$, $\nabla g(x,y) = (2x, -2y)$

$$\nabla f = \lambda \nabla g \Rightarrow \begin{cases} 1 = \lambda 2x \\ -1 = -\lambda 2y \end{cases}$$

Hence $\begin{cases} \lambda 2x = 1 \\ -\lambda 2y = -1 \Rightarrow \lambda y = 1/2 = \lambda x \\ x^2 - y^2 = 2 \end{cases}$, note $\lambda \neq 0$

$\Rightarrow y = x$ also $2\lambda x = 0 \neq 1$

But $x^2 - y^2 = 0 \neq 2$ for $x = y$, and so the Lagrange theorem fails and there is no max/min on the given constraint. ✓

6.) $f(x,y,z) = x + y + z$, $g_1(x,y,z) = x^2 - y^2 = 1$

$g_2(x,y,z) = 2x + z = 1$

$\nabla f = (1,1,1)$

$\nabla g_1 = (2x, -2y, 0)$, $\nabla g_2 = (2,0,1)$

$\Rightarrow \nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2$

$\Rightarrow (1,1,1) = \lambda_1 (2x, -2y, 0) + \lambda_2 (2,0,1)$

$\Rightarrow$

$$\begin{cases} 2\lambda_1 x + 2\lambda_2 = 1 \\ -2\lambda_1 y = \cancel{0}\ 1 \\ \lambda_2 = 1 \\ x^2 - y^2 = 1 \\ 2x + z = 1 \end{cases}$$

$\Rightarrow 2\lambda_1 x + 2 = 1$

$\Rightarrow 2\lambda_1 x = -1$

$\Rightarrow \lambda_1 x = -1/2 \Rightarrow \lambda_1 \neq 0$

$\Rightarrow -2\lambda_1 y = \cancel{0}\ 1 \Rightarrow y = \frac{-1}{2\lambda_1} \Leftrightarrow \lambda_1 = \frac{-1}{2y}, y \neq 0$

10.5

$$\Rightarrow \cancel{2}\left(\frac{-1}{\cancel{2}y}\right)x + 2 = 1$$

$$\Rightarrow \frac{-x}{y} = -1 \Rightarrow x = y \;\checkmark$$

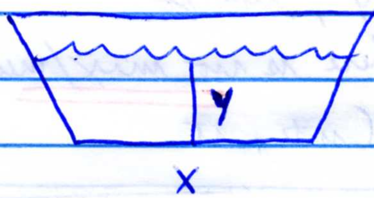
But $x^2 - y^2 = 1 \Rightarrow x^2 - (x^2) = 0 \neq 1$ CONTRADICTION!

Hence there is no max/min of $f(x,y,z) = x+y+z$ on the given restrictions.

~~$\Rightarrow z = 1 \mp 2 \Rightarrow z = -1$ or $z = 3$~~

~~Hence $\boxed{(1, 0, -1)}$ and $\boxed{(-1, 0, 3)}$ are the extrema~~ ✓

26.)



$A = y(x + y\tan\theta)$

$P = x + 2y/\cos\theta$

$\nabla P(x,y) = \left(1, \frac{2}{\cos\theta}\right)$

$\nabla A(x,y) = (y, x + 2\tan\theta\, y)$

$$\nabla P = \lambda \nabla A \Rightarrow \begin{cases} 1 = \lambda y \Rightarrow \quad y \neq 0,\ \lambda \neq 0 \\ \frac{2}{\cos\theta} = \lambda(x + 2\tan\theta\, y) \\ y(x + y\tan\theta) = A \end{cases}$$

$$\Rightarrow \frac{1}{\lambda} = y \Rightarrow \frac{2}{\lambda\cos\theta} = (x + 2\tan\theta\, y)$$

$$\Rightarrow \frac{2y}{\cos\theta} = (x + 2\tan\theta\, y) \checkmark$$

$$\Rightarrow x = \frac{2y}{\cos\theta} - 2\tan\theta\, y$$

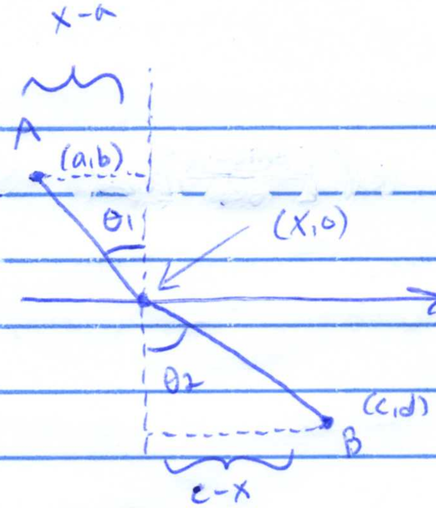
$$= 2y\left[\frac{1 - \sin\theta}{\cos\theta}\right] \checkmark$$

$$\Rightarrow y\left(2y\left[\frac{1-\sin\theta}{\cos\theta}\right] + \tan\theta\, y\right) = A$$

$$\Rightarrow y^2\left[\frac{2(1-\sin\theta)}{\cos\theta} + \frac{\sin\theta}{\cos\theta}\right] = A$$

$$\Rightarrow y^2\left[\frac{2-\sin\theta}{\cos\theta}\right] = A \Rightarrow y^2 = \frac{A\cos\theta}{2-\sin\theta} \checkmark$$

28.)



note: $\sin\theta_1 = \sqrt{(a-x)^2+b^2} / (x-a)$

$\sin\theta_2 = \sqrt{(c-x)^2+d^2} / (c-x)$

distance traveled is $\underbrace{\sqrt{(a-x)^2+b^2}}_{d_1} + \underbrace{\sqrt{(c-x)^2+d^2}}_{d_2}$

time taken:

$d_1 v_1^{-1} + d_2 v_2^{-1}$

$$T(x) = \frac{1}{v_1}\sqrt{(a-x)^2+b^2} + \frac{1}{v_2}\sqrt{(c-x)^2+d^2} = \text{total time}$$

$$\frac{dT}{dx} = \frac{1}{\cancel{2}v_1}\left((a-x)^2+b^2\right)^{-1/2}(-\cancel{2})(a-x) + \frac{1}{\cancel{2}v_2}\left((c-x)^2+d^2\right)^{-1/2}(-\cancel{2})(c-x)$$

$$= \frac{(x-a)}{v_1\sqrt{(a-x)^2+b^2}} + \frac{(x-c)}{v_2\sqrt{(c-x)^2+d^2}} = 0$$

$$\Rightarrow \frac{x-a}{v_1\sqrt{(a-x)^2+b^2}} = \frac{-(x-c)}{v_2\sqrt{(c-x)^2+d^2}}$$

$$\Rightarrow \underbrace{\frac{x-a}{-x+c}\,\frac{\sqrt{(c-x)^2+d^2}}{\sqrt{(a-x)^2+b^2}}}_{} = \frac{v_1}{v_2} \checkmark$$

$$= \frac{\sin\theta_1}{\sin\theta_2} \checkmark$$