MATH 202: SOLUTIONS to HW #2

1.5

(10) $A = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$AB = \begin{bmatrix} 3 & 0 & 1 \\ 1 & 2 & -1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -3 \\ 5 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix}$$

$$\det A = 3 \cdot \det \begin{bmatrix} 2 & -1 \\ 0 & 1 \end{bmatrix} - 0 \cdot \det \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} + 1 \cdot \det \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$$
$$= 6 - 2 = 4$$

$$\det B = 1 \cdot \det \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - 0 \cdot \det \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix} + (-1) \cdot \det \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$$
$$= -1 - 2 = -3$$

$$\det AB = 3 \cdot \det \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} - 1 \cdot \det \begin{bmatrix} 5 & 1 \\ 1 & -1 \end{bmatrix} + (-3) \cdot \det \begin{bmatrix} 5 & -1 \\ 1 & 1 \end{bmatrix}$$
$$= 0 + 6 - 18 = -12 \quad \left[= \det A \cdot \det B \;!!! \right]$$

$$\det(A+B) = \det \begin{bmatrix} 4 & 0 & 0 \\ 3 & 2 & 0 \\ 1 & 1 & 1 \end{bmatrix} = 8 \quad \left[\neq \det A + \det B \;!!! \right]$$

(21) The inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

(NOTICE that $ad-bc = \det\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, which we are assuming to be nonzero. This problem shows computationally that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is invertible $\Longleftrightarrow \det \neq 0$!)

proof: A^{-1} is the inverse of $A \Longleftrightarrow A^{-1}A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so we just need to check that this holds:

$$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \frac{1}{ad-bc}\begin{bmatrix} ad-bc & bd-bd \\ -ac+ac & -bc+ad \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{by cancellation.} \qquad \square$$

(22) We are interested in finding solutions to the system of equations:

$$\begin{cases} ax + by = e \\ cx + dy = f \end{cases}$$

Notice that $\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} ax + by \\ cx + dy \end{bmatrix}$,

so we can re-write the equations as an equality of column vectors:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}.$$

In other words, the pair (x, y) satisfies the original pair of equations if and only if the matrix/linear function $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ applied to $\begin{bmatrix} x \\ y \end{bmatrix}$ is the vector $\begin{bmatrix} e \\ f \end{bmatrix}$.

In the last exercise, we showed that the inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the matrix $\frac{1}{ad - bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Therefore, (x, y) is a solution to the pair of equations if and only if

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} e \\ f \end{bmatrix}$$

$$\iff \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix}$$

$$\iff \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \cancel{\begin{bmatrix} x \\ y \end{bmatrix}} \begin{bmatrix} e \\ f \end{bmatrix}$$

$$\iff \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \begin{bmatrix} e \\ f \end{bmatrix}$$

(24) Let $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

Then:

$$AB = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$$

and

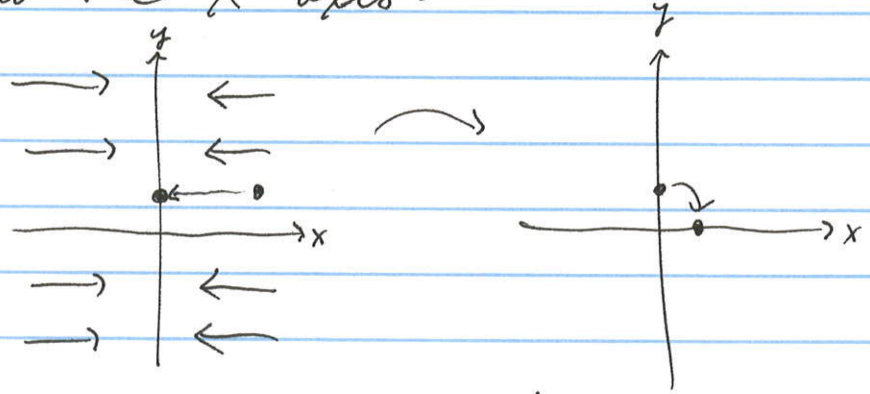
$$BA = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq 0$$

Geometrically, A represents the linear function

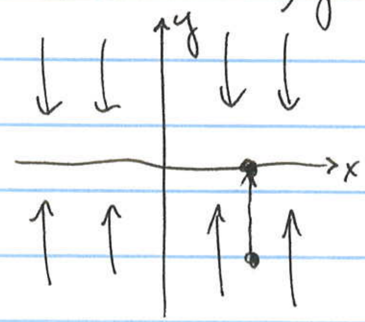
$$(x, y) \longmapsto (y, 0)$$

which ~~takes the y-axis to the x-axis and~~ projects onto the y-axis then sends the y-axis to the x-axis:

A:



B represents the linear function which projects onto the x-axis: $(x, y) \longmapsto (x, 0)$



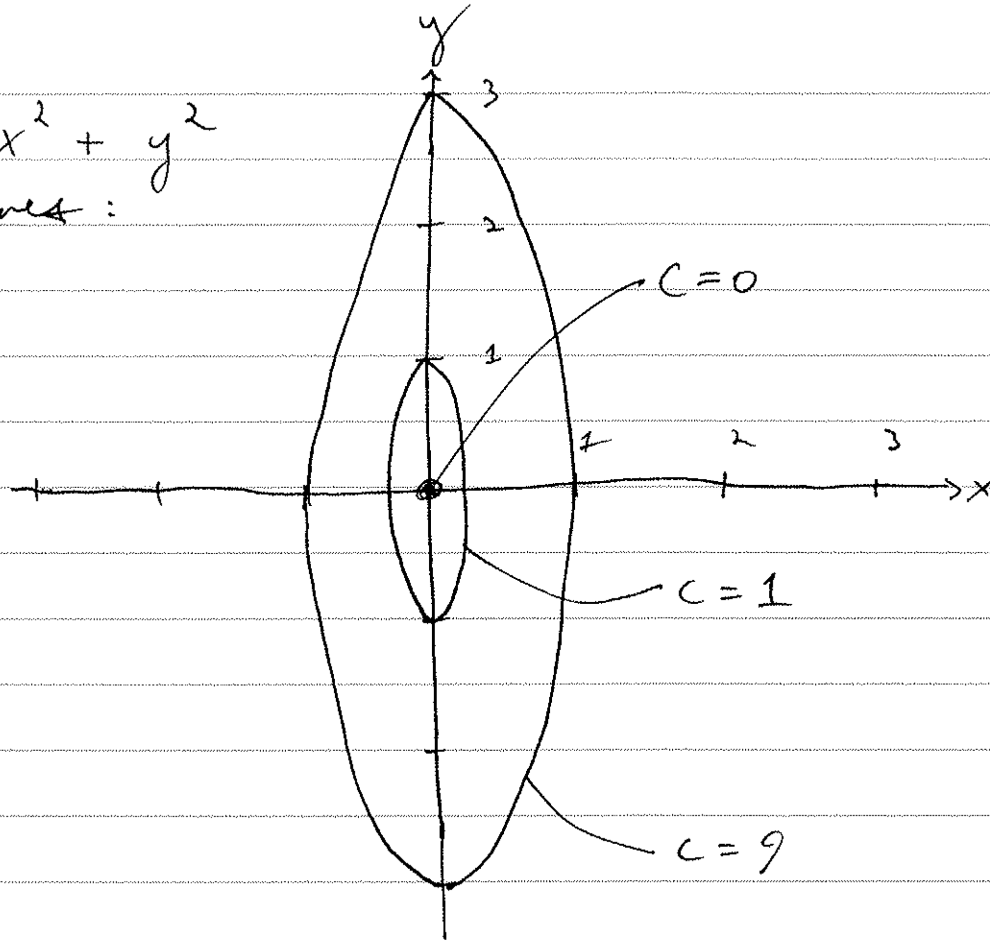
Geometrically, AB projects onto the x-axis, then the y-axis, which forces everything to be sent to $(0, 0)$.

BA projects onto the y-axis, rotates to the x-axis, then projects onto the x-axis (which has no effect b/c the point is already there).

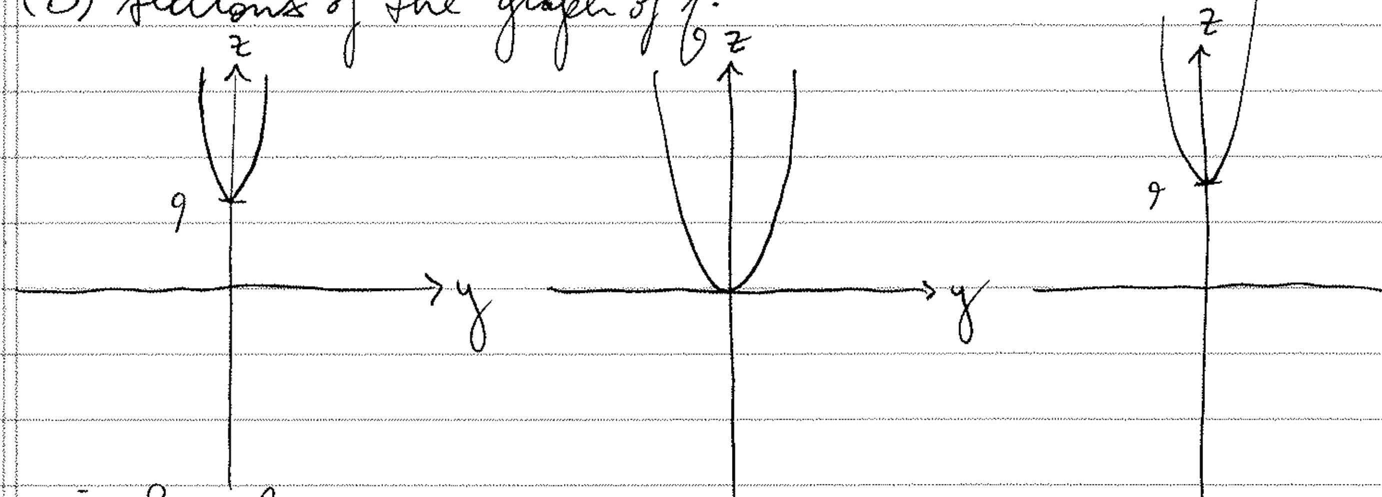
2.1

(6) $f(x,y) = 9x^2 + y^2$

(A) level curves:



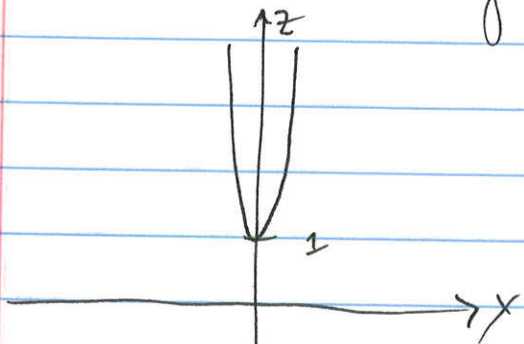
(B) sections of the graph of f:



in the plane $x=-1$: $\quad$ in the plane $x=0$: $\quad$ in the plane $x=-1$:

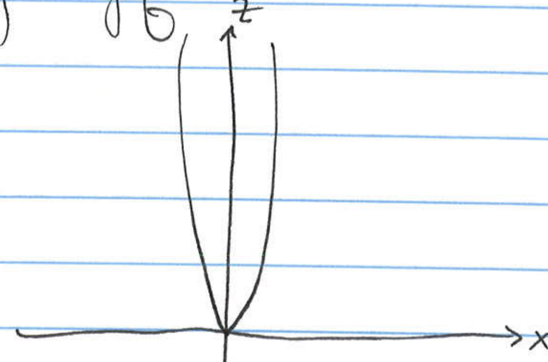
$z = 9 + y^2$ $\quad$ $z = y^2$ $\quad$ $z = 9 + y^2$

(c) the sections of the graph of f:



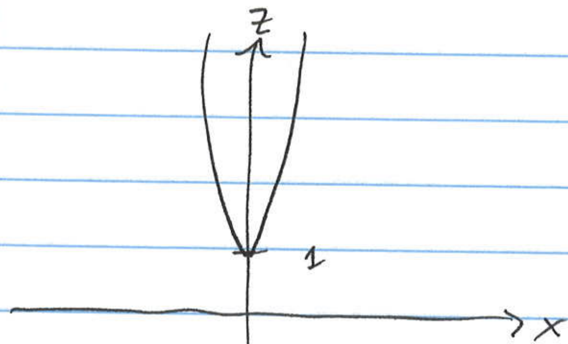
in the plane $y = -1$:

$z = 9x^2 + 1$



in the plane $y = 0$:

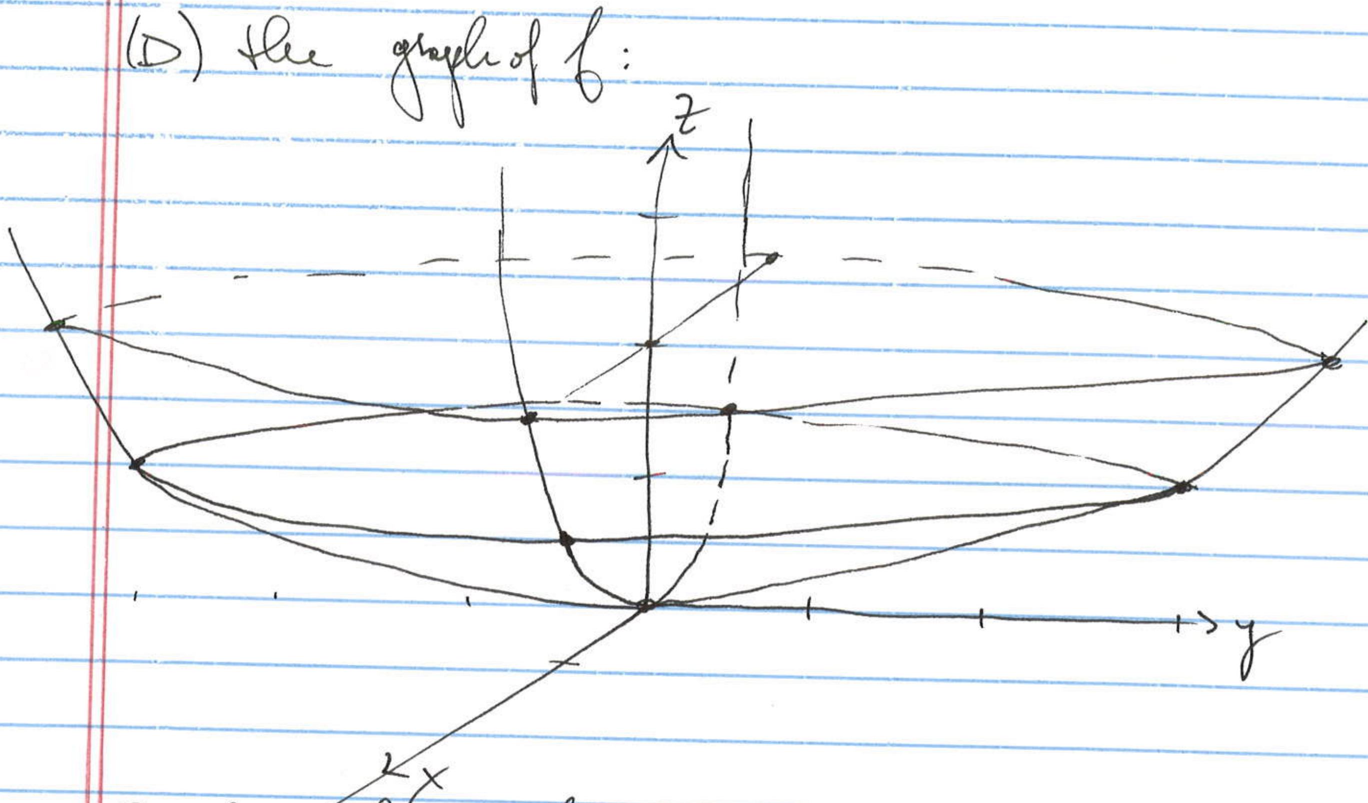
$z = 9x^2$



in the plane $y = 1$:

$z = 9x^2 + 1$

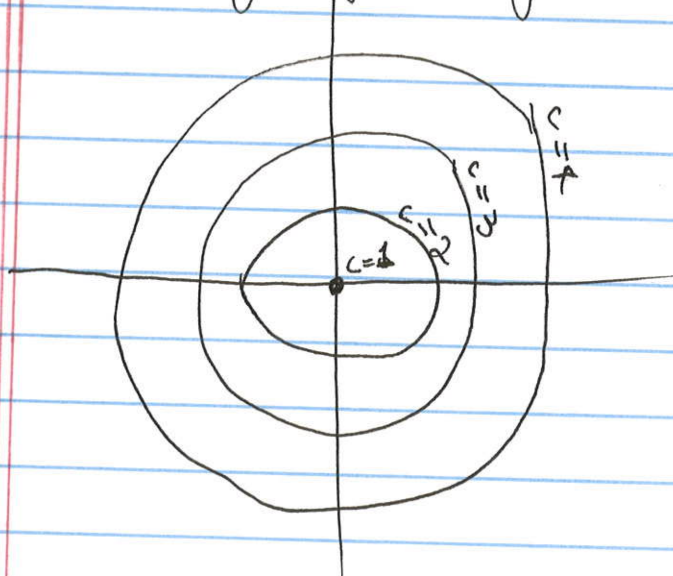
(D) the graph of f:



Broadly-shaped bowl sort of object, stretched by a factor of 3 in the y-direction.

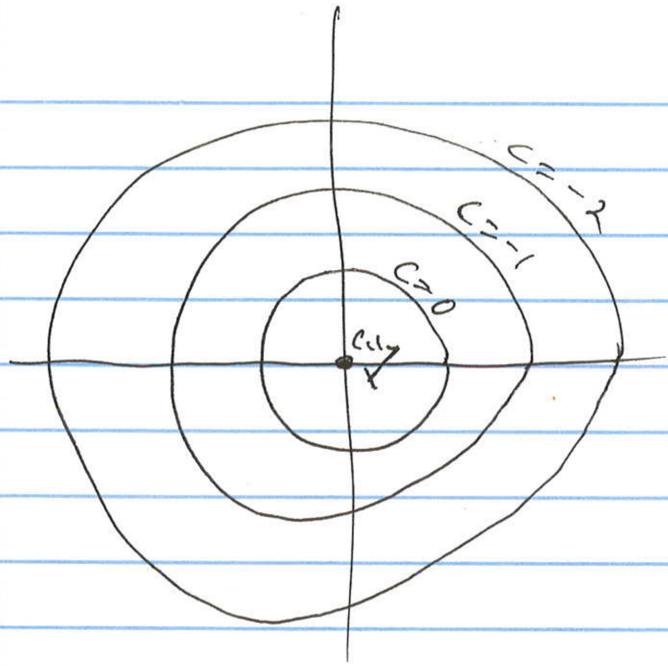
(10) (A) $f(x,y) = x^2 + y^2 + 1$

For $c < 1$, $f(x,y) = c$ has no solutions because $x^2 + y^2 \geq 0$.
For $c = 1$, the level set $f(x,y) = 1$ is the origin and for $c > 1$, the level sets are circles of expanding radii.



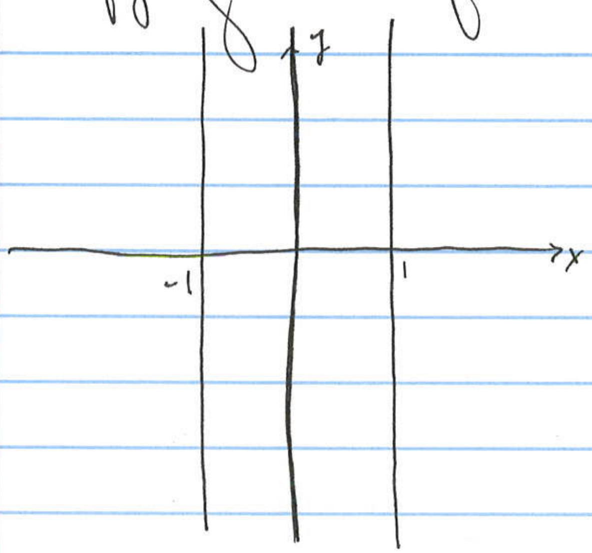
(B) $f(x,y) = 1 - x^2 - y^2$

For $c > 1$, $f(x,y) = c$ has no solutions because $-x^2 - y^2 \leq 0$.
For $c = 1$, the level set $f(x,y) = 1$ is the origin and for $c < 1$ the level sets are circles of expanding radii.

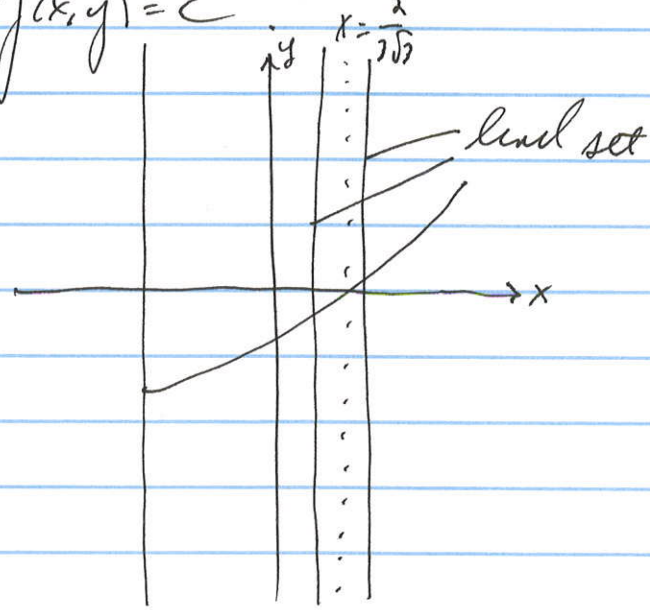


(c) $f(x,y) = x^3 - x$.

The function does not depend on y, so the level sets will be vertical lines for each x-value satisfying the equation $f(x,y) = c$.

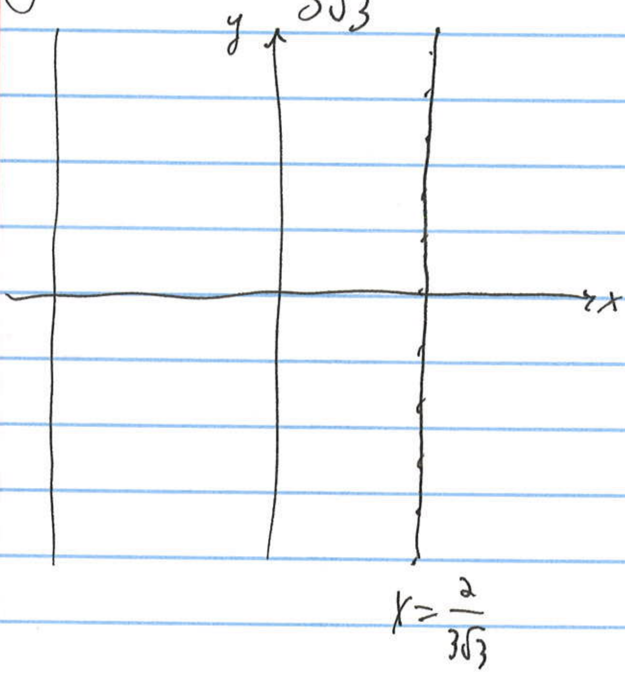


$c = 0$
the vertical lines
$x = 0, -1, 1$



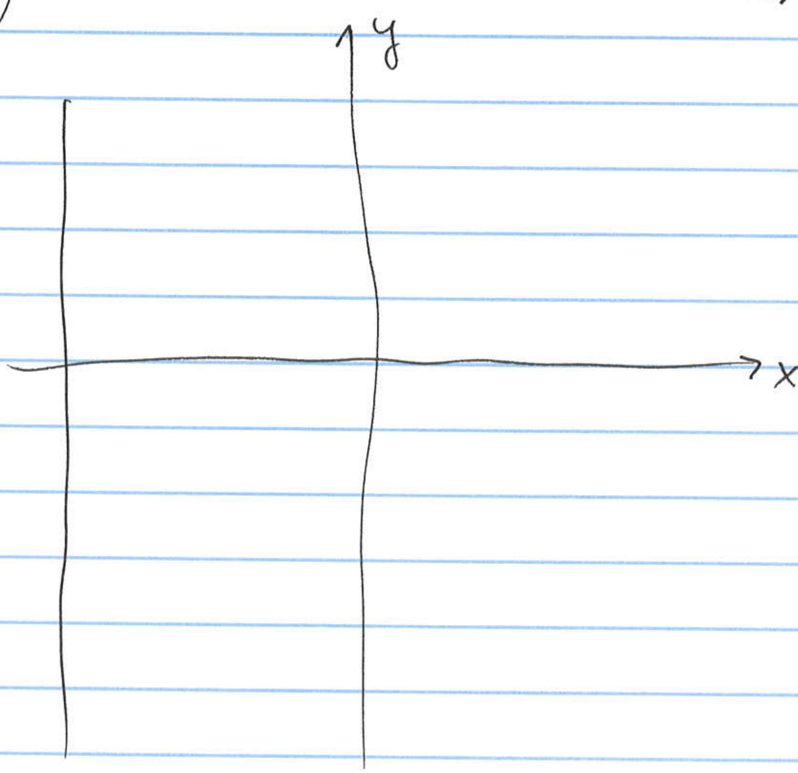
for $-\frac{2}{3\sqrt{3}} < c < \frac{2}{3\sqrt{3}}$,
three vertical lines

for $c = \pm \frac{2}{3\sqrt{3}}$, two vertical lines:

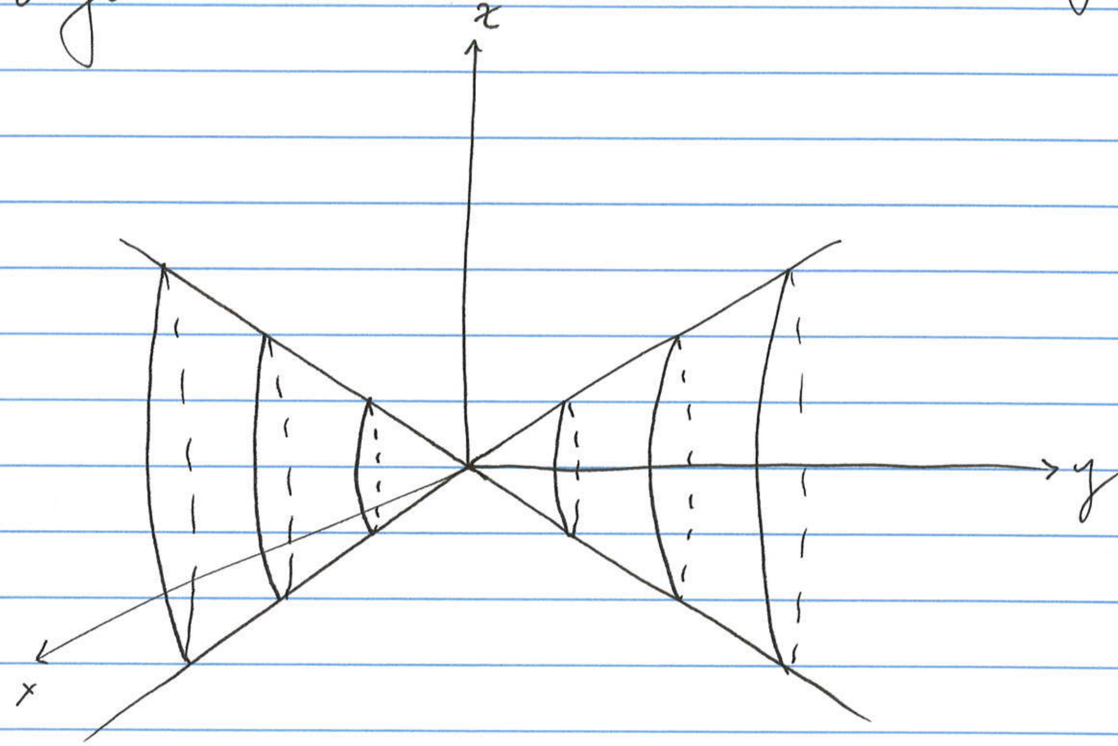


(this case corresponds to the inflection point of $z = x^3 - x$ being translated up/down to meet the origin.)

for $c < -\frac{2}{3\sqrt{3}}$ OR $c > \frac{2}{3\sqrt{3}}$, a single vertical line



(36) The surface defined by the equation $y^2 = x^2 + z^2$ is a pair of cones with rotational symmetry around the y-axis, with their tips meeting at the origin:



2.2

(6) $$f(x,y) = \begin{cases} \dfrac{xy^3}{x^2+y^6} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$$

(A) $$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y\text{-axis} \\ (x=0)}} f(x,y) = \lim_{y\to 0} \frac{0\cdot y^3}{0^2+y^6} = \lim_{y\to 0} 0 = 0$$

(B) $$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } x=y^3}} f(x,y) = \lim_{y\to 0} \frac{y^3\cdot y^3}{(y^3)^2+y^6} = \lim_{y\to 0} \frac{y^6}{2y^6} = \frac{1}{2}$$

(C) f is NOT continuous at $(0,0)$ because $\lim_{(x,y)\to(0,0)} f(x,y)$ DOES NOT EXIST (it has different values for different paths to $(0,0)$.)

In particular $\lim_{(x,y)\to(0,0)} f(x,y) \neq 0 = f(0,0)$,

so f is NOT CONTINUOUS at $(0,0)$.

(8) (A) By algebraic manipulation,

$$\frac{(x+y)^2-(x-y)^2}{xy} = \frac{x^2+2xy+y^2-x^2+2xy-y^2}{xy}$$

$$= \frac{4xy}{xy} = 4.$$

Therefore,

$$\lim_{(x,y)\to(0,0)} \frac{(x+y)^2-(x-y)^2}{xy} = \lim_{(x,y)\to(0,0)} 4 = 4$$

(B) We will prove that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{\sin xy}{y} = 0.$

One way to make a heuristic calculation is to use the Taylor series for the sine function:

$$\sin t = t - \frac{t^3}{3!} + \frac{t^5}{5!} - \frac{t^7}{7!} + \cdots$$

Letting $t = xy$, we get:

$$\frac{\sin xy}{y} = \frac{1}{y}\left(xy - \frac{(xy)^3}{3!} + \frac{(xy)^5}{5!} - \cdots\right)$$

$$= x - \frac{x^3y^2}{3!} + \frac{x^5y^4}{5!} - \cdots$$

$$\longrightarrow 0 \quad \text{as} \quad (x,y)\to(0,0)$$

Here is an ε/δ proof of this fact.

proof: Let $\varepsilon > 0$ and set $\delta = \varepsilon$.

Suppose that $0 < \|(x,y) - (0,0)\| < \delta$. Then

$$|x| \le \sqrt{x^2+y^2} = \|(x,y)\| < \delta.$$

We will use the fact that $|\sin t| \le |t|$ for all $t \in \mathbb{R}$. This can be proved using the Taylor series for $\sin t$, by deriving it from the fact $\lim_{t\to 0} \frac{\sin t}{t} = 1$ (for small t) or from any suitable definition of the sine function. Since this means that $|\sin xy| \le |xy|$, we find that:

$$\left|\frac{\sin xy}{y} - 0\right| = \frac{|\sin xy|}{|y|} \le \frac{|xy|}{|y|}$$

$$= \frac{|x||y|}{|y|}$$

$$= |x| < \delta = \varepsilon.$$

as desired. $\square$

(8)(c) I will first prove that $\lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} = 0$.

Let $\varepsilon > 0$. Set $\delta = \varepsilon$, and suppose that

$0 < \|(x,y)-(0,0)\| < \delta$.

Then $|x| \le \sqrt{x^2+y^2} = \|(x,y)\| < \delta$. We now find that:

$$\left| \frac{x^3}{x^2+y^2} - 0 \right| = \frac{|x|^3}{|x^2+y^2|} \le \frac{|x|^3}{|x|^2} \quad \left(\text{since } x^2 \le x^2+y^2, \ \frac{1}{|x^2+y^2|} \le \frac{1}{|x^2|} \right)$$

$$= |x| < \delta = \varepsilon. \qquad \square$$

By a very similar proof, $\lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2} = 0$.

Using the linearity properties of the limit, we find:

$$\lim_{(x,y)\to(0,0)} \frac{x^3 - y^3}{x^2+y^2} = \lim_{(x,y)\to(0,0)} \frac{x^3}{x^2+y^2} - \lim_{(x,y)\to(0,0)} \frac{y^3}{x^2+y^2}$$

$$= 0 - 0 = 0$$

(12) (A) We will use l'Hôpital's rule for single variable functions:

If $\frac{f(x)}{g(x)} \to \frac{0}{0}$ or $\frac{\infty}{\infty}$ as $x \to a$,

then $\lim\limits_{x \to a} \frac{f(x)}{g(x)} = \lim\limits_{x \to a} \frac{f'(x)}{g'(x)}$

The limit in question is

$\lim\limits_{x \to 0} \frac{\sin 2x - 2x}{x^3}$. (The fraction tends to $\frac{0}{0}$ as $x \to 0$ so:)

$\overset{\text{l'Hôp}}{=} \lim\limits_{x \to 0} \frac{2\cos 2x - 2}{3x^2}$ (again this tends to $\frac{0}{0}$ as $x \to 0$ so:)

$\overset{\text{l'Hôp}}{=} \lim\limits_{x \to 0} \frac{-4\sin 2x}{6x}$ (again this tends to $\frac{0}{0}$ as $x \to 0$ so:)

$\overset{\text{l'Hôp}}{=} \lim\limits_{x \to 0} \frac{-8\cos 2x}{6} = -\frac{8}{6} = -\frac{4}{3}$

(12) (B) ~~First~~ we will take the limit for $(x,y) = (x,0)$

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along}\\ x\text{-axis}}} \frac{\sin 2x - 2x + y}{x^3 + y} = \lim_{x\to 0} \frac{\sin 2x - 2x}{x^3} = -\frac{4}{3}$$

by part (A)

Next we will take the limit along the curve $y = 2x$:

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along}\\ y=2x}} \frac{\sin 2x - 2x + y}{x^3 + y} = \lim_{x\to 0} \frac{\sin 2x - 2x + 2x}{x^3 + (2x)}$$

$$= \lim_{x\to 0} \frac{\sin 2x}{2x} \cdot \frac{2}{x^2 + 2}$$ (clever algebraic manipulation)

$$= \lim_{x\to 0} \frac{\sin 2x}{2x} \cdot \lim_{x\to 0} \frac{2}{x+2}$$ (split limits over $\cdot$)

$$= \lim_{x\to 0} \frac{2\cos 2x}{2} \cdot 1$$ (l'Hôpital's rule and direct evaluation of continuous function)

$$= 1 \cdot 1 = 1$$

Since $-\frac{4}{3} \neq 1$, the limit as $(x,y)\to(0,0)$ DOES NOT EXIST.

(12) (c) In the book they prove that

$$\lim_{(x,y)\to(0,0)} \frac{2x^2y}{x^2+y^2} = 0$$

Since $-1 \le \cos z \le 1$, we could use the squeeze lemma to argue that

$$\lim_{(x,y,z)\to(0,0,0)} -\frac{2x^2y}{x^2+y^2} \le \lim_{(x,y,z)\to(0,0,0)} \frac{2x^2y\cos z}{x^2+y^2} \le \lim_{(x,y,z)\to(0,0,0)} \frac{2x^2y}{x^2+y^2}$$

and so: $0 \le \lim_{(x,y,z)\to(0,0,0)} \frac{2x^2y\cos z}{x^2-y^2} \le 0$,

which shows that $\lim_{(x,y,z)\to(0,0,0)} \frac{2x^2y\cos z}{x^2-y^2} = 0$

This is a valid proof. We will also give a direct ε/δ proof in the spirit of the proof in the book.

To prove that $\lim_{(x,y,z)\to(0,0,0)} \frac{2x^2y\cos z}{x^2-y^2} = 0$,

Let $\varepsilon > 0$ and set $\delta = \varepsilon/2$.

Suppose that $0 < \|(x,y,z)-(0,0,0)\| < \delta$.

Then $|y| \le \sqrt{x^2+y^2+z^2} = \|(x,y,z)\| < \delta$.

Since $|\cos z| < 1$, we may conclude that:

$$\left|\frac{2x^2y\cos z}{x^2+y^2}\right| = \left|\frac{2x^2y}{x^2+y^2}\right| |\cos z|$$

$$\le \left|\frac{2x^2y}{x^2+y^2}\right| \cdot 1$$

$$\le \left|\frac{2x^2y}{x^2}\right| \qquad \left(\text{since } x^2 \le x^2+y^2,\ \frac{1}{x^2+y^2} \le \frac{1}{x^2}\right)$$

$$= 2|y|$$

$$< 2\delta = 2 \cdot \frac{\varepsilon}{2} = \varepsilon. \qquad \square$$

(14) Let $f(x,y,z) = \dfrac{1}{x^2+y^2+z^2-1}$

If $x_0^2+y_0^2+z_0^2 \neq 1$, then

$$\lim_{(x,y,z)\to(x_0,y_0,z_0)} f(x,y,z) = \lim_{(x,y,z)\to(x_0,y_0,z_0)} \frac{1}{x^2+y^2+z^2-1}$$

$$= \frac{1}{x_0^2+y_0^2+z_0^2-1} = f(x_0,y_0,z_0),$$

so f is continuous at (x_0,y_0,z_0).

If $x_0^2+y_0^2+z_0^2 = 1$, then $f(x_0,y_0,z_0)$ is NOT defined so in particular f is NOT continuous at (x_0,y_0,z_0).

Therefore the points where f is discontinuous are all (x,y,z) satisfying

$$x^2+y^2+z^2 = 1,$$

i.e. the sphere centered at the origin of radius 1.

(26) Using the change of coordinates to spherical coordinates

$$\begin{cases} x = \rho \sin\phi \cos\theta \\ y = \rho \sin\phi \sin\theta \\ z = \rho \cos\phi \end{cases}$$

(note that $x^2+y^2+z^2=\rho^2$)

as $(x,y,z) \to (0,0,0)$ the variables θ, ϕ are not constrained, but the radius $\rho \to 0$.

Therefore,

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xyz}{x^2+y^2+z^2} = \lim_{\rho\to 0} \frac{(\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)}{\rho^2}$$

$$= \lim_{\rho\to 0} \frac{\rho^3}{\rho^2} \sin^2\phi \cos\phi \cos\theta \sin\theta$$

$$= \lim_{\rho\to 0} \rho \sin^2\phi \cos\phi \cos\theta \sin\theta = 0$$